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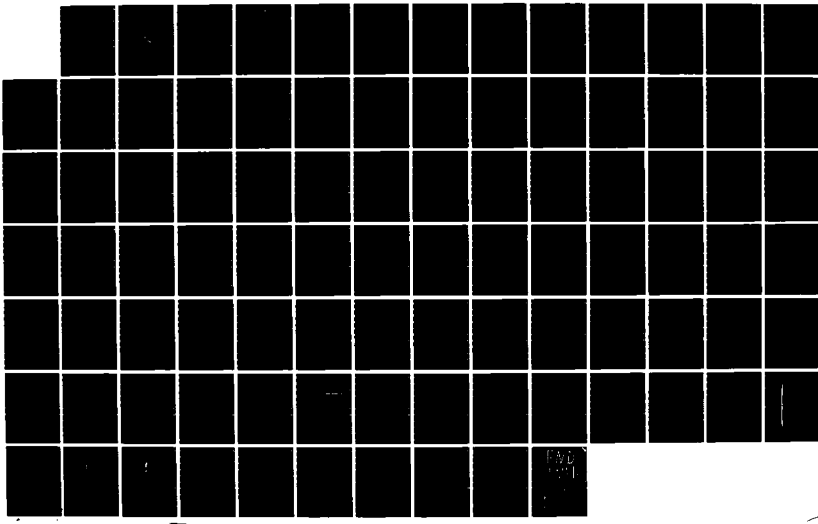

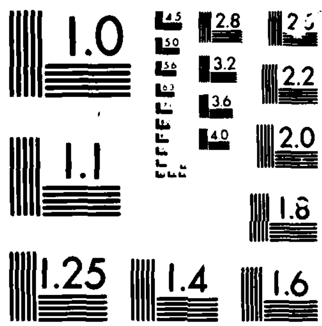

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EQUATIONS OF LAMINAR AND TURBULENT FLOWS IN GENERAL CURVILINEAR COORDINATES

by

M. C. Richmond, H. C. Chen, and V. C. Patel

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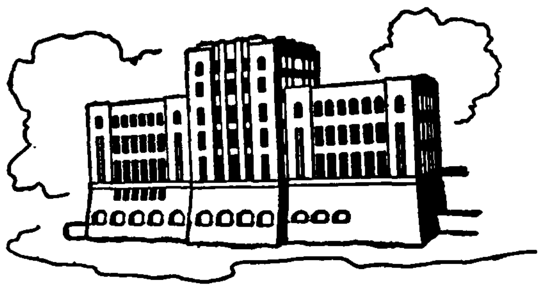

IIHR Report No. 300

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Recent developments in computational fluid mechanics and grid generation have made it possible to consider complex three-dimensional laminar and turbulent flows using nonorthogonal coordinates. However, the necessary equations in generalized nonorthogonal coordinates are not readily available in a single reference. The purpose of this report is to present these equations for an

incompressible fluid. Methods of curvilinear tensor analysis are used to develop the most commonly used equations. Their relationship with the more familiar equations in orthogonal coordinates is shown. Finally, the practical problems of evaluating the many geometric coefficients which appear in these equations are examined by considering numerically-generated coordinates for flow around three-dimensional bodies. Keywords:

CONTENTS

ABSTRACT

Recent developments in computational fluid mechanics and grid generation have made it possible to consider complex three-dimensional laminar and turbulent flows using nonorthogonal coordinates. However, the necessary equations in generalized nonorthogonal coordinates are not readily available in a single reference. The purpose of this report is to present these equations for an incompressible fluid. Methods of curvilinear tensor analysis are used to develop the most commonly used equations. Their relationship with the more familiar equations in orthogonal coordinates is shown. Finally, the practical problems of evaluating the many geometric coefficients which appear in these equations are examined by considering numerically-generated coordinates for flow around three-dimensional bodies.

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LIST OF SYMBOLS

$A^i, A_i, A(i)$	contravariant, covariant, and physical components of a general vector $\underset{\sim}{A}$
$A^i_{,j}$, $A_{i,j}$	covariant derivatives of A^i and A_i, defined in (2-58, 2-57)
$A(i,j)$	physical component of $A^i_{,j}$
$\underset{\sim}{a}^i$, $\underset{\sim}{a}_i$	contravariant and covariant base vectors for general curvilinear coordinates, defined in (2-12,2-15)
C(ims), D(in), E(im) P(i), Q(is)	physical geometric coefficients appearing in the covariant derivative and Laplacian of a vector, defined in (A-15)
C_μ, $C_{\varepsilon 1}$, $C_{\varepsilon 2}$	turbulence model constants
$\frac{DA}{Dt}$	material derivative of A, defined in (3-1)
e	internal energy
e_{ij}, e^{ij}	covariant and contravariant strain rate tensors, defined in (3-25, 3-28)
$\underset{\sim}{e}^i$, $\underset{\sim}{e}_i$	contravariant and covariant unit base vectors for general curvilinear coordinates, defined in (2-22a,b)
f(m)	physical geometric coefficient appearing in the Laplacian of a scalar, defined in (A-14)
g_{ij}, g^{ij}	covariant and contravariant general metric tensors, defined in (2-29c, 2-34)
g(ij)	physical components of g^{ij}
g	determinant of g_{ij}
G	turbulence generation term
h_i	metric scale factor for orthogonal coordinates
J	Jacobian determinant, defined in (2-3)
K_{ij}	curvature parameters for orthogonal coordinates, defined in (6-3)
k	turbulent kinetic energy
$\underset{\sim}{n}$	outward normal vector
p	instantaneous or mean pressure

p'	fluctuating pressure
$\underset{\sim}{q}$	heat flux vector
$\underset{\sim}{r}$	position vector
$\underset{\sim}{t}$	stress vector, defined in (3-13)
V^i, $V(i)$	instantaneous or mean contravariant and physical components of the velocity vector $\underset{\sim}{V}$
$\overline{V^i}$	(1) quantity in transformed coordinates (2) Reynolds averaged quantity
$\mathcal{V}$	volume
x^i	general curvilinear or cartesian coordinates (i=1,2,3)
Γ^i_{jk}	Christoffel symbols, defined in (2-54)
$\Gamma(ijk)$	physical components of Γ^i_{jk}, defined in (2-59c)
δ^i_j	Kronecker delta symbol, defined in (2-17)
δ_{ij}	Euclidean metric tensor, defined in (2-28)
ε	rate of turbulent energy dissipation
$\varepsilon_{\ell mn}$, $\varepsilon^{\ell mn}$	alternating tensors, defined in (2-20, 2-33)
λ	second viscosity coefficient
μ	dynamic viscosity
ν	kinematic viscosity
ν_t	turbulent eddy viscosity, defined in (5-3)
$\nu_{t,n}$	covariant derivative of ν_t
ξ^i, η^i	general curvilinear coordinates (i=1,2,3)
ρ	fluid density
σ^{ij}	stress tensor, defined in (3-18)
σ_ε, σ_k	turbulence model constants
τ^{ij}	deviatoric stress tensor, defined in (3-28)
Φ	dissipation function, defined in (3-44)
ϕ	a general scalar function

$\nabla\phi$	gradient of ϕ
$\nabla^2\phi$	Laplacian of a scalar, defined in (A-13)
$\nabla^2\underset{\sim}{V}$	Laplacian of a vector, defined in (A-15)
$\frac{\partial}{\partial\xi(m)}$	$g_{mm}^{-1/2}\frac{\partial}{\partial\xi^m}$, physical partial derivative

I. INTRODUCTION

A problem that frequently arises in the development and application of general numerical methods for the solution of fluid-flow equations is the choice of a coordinate system. This problem becomes particularly crucial in the analysis of three-dimensional flows, whether internal or external, since the coordinates chosen must satisfy many conflicting requirements and influence the accuracy of the solutions. For example, the application of boundary conditions on a solid wall is greatly facilitated by locally-orthogonal body-fitted coordinates, while proper resolution of the flow in certain regions necessitates the use of nonorthogonal curvilinear coordinates.

Many methods are now available for the construction of general nonorthogonal coordinates, but their use in the actual solutions of fluid-flow problems involving three-dimensional geometries is still in its infancy. Once a coordinate system is selected for a given geometry, or class of problems, there still remains the task of transforming the equations of motion from the familiar orthogonal coordinates to those in the chosen system. Two different approaches have been adopted for this purpose. One of these uses partial transformations, in which only the independent coordinate variables are transformed, leaving the dependent variables (i.e. the velocity components) in the original orthogonal coordinates. This approach, which has been used by Chen and Patel (1985a, 1985b), among others, has the advantages that the resulting equations are relatively simple and the results can be readily interpreted. If the physical geometry is highly curved, such an approach may lead to increased numerical diffusion due to large skew angles between the velocity components and the faces of the computational cell. The alternative is to transform the equations completely, including the dependent as well as the independent variables. This approach has been used principally in two-dimensional flow problems since, in three dimensions, the equations involve many more geometric coefficients. This leads to increased storage requirements if the coefficients are calculated once and for all, or longer execution times if they are calculated each time they are needed. It is also possible that numerical errors associated with the evaluation of the geometric coefficients could adversely affect the solution of the flow equations. However,

use of a complete transformation provides a method to obtain solutions for a much wider variety of problems than is possible with the partially-transformed equations.

In spite of the rapid developments of methods for the construction of generalized, nonorthogonal, curvilinear coordinates for fluid flows with three-dimensional geometries (see, for example, Thompson, 1982), there does not appear to be a ready reference in which one can find the corresponding equations for laminar and turbulent flows. In most instances, the equations are derived as needed for a particular coordinate system and it is usually quite difficult to make generalizations or extensions.

The partial transformation of the Reynolds-averaged Navier-Stokes equations, and the transport equations of the well known k-ϵ model of turbulence, from a general curvilinear orthogonal coordinate system to a nonorthogonal system were presented by Chen and Patel (1985a). The main purpose of the present report is to document, for general reference, the fully-transformed equations. Because methods of tensor analysis are used throughout the report, a summary of the basic concepts of tensor analysis is presented first. The Reynolds-averaged Navier-Stokes equations and the transport equations for the Reynolds stresses are then developed for an incompressible fluid without reference to any particular turbulence model. The equations of turbulent kinetic energy and isotropic dissipation, which form the basis of the k-ϵ model, are then presented. The interrelationship between coordinates is then demonstrated by reducing the equations in generalized coordinates to those given by Nash and Patel (1972) for orthogonal coordinates. Finally, to provide practical examples, typical geometric coefficients that appear in the equations are evaluated from numerically-generated nonorthogonal coordinates for two ship-like three-dimensional bodies.

II. ELEMENTS OF TENSOR ANALYSIS

The tools of curvilinear tensor analysis will be used extensively throughout this report because they provide the most direct method by which equations valid for nonorthogonal coordinate systems may be derived. The texts by Lass (1950) and Aris (1962) are the main references for the material concerning curvilinear tensors presented herein. This section provides a summary of useful concepts and expressions frequently used in the later sections and which the reader may find useful in the application or extension of the results presented in this report.

II.1. Coordinate Transformations

Curvilinear tensor analysis differs fundamentally from Cartesian tensor analysis in that the base vectors (unit vectors) can be functions of position. General curvilinear coordinates are denoted by $x^i = (x^1, x^2, x^3)$ and $\xi^i = (\xi^1, \xi^2, \xi^3)$. The coordinates can be defined either analytically or numerically and may be, in general, orthogonal or nonorthogonal. The following summation convention is used throughout this report: if an index appears in a product of terms once as a subscript and once as a superscript a summation upon the index from 1 to n, where n is the dimension of the space, is implied. For example, in a three-dimensional space

$$A^i A_i = A^1 A_1 + A^2 A_2 + A^3 A_3$$

where i is a dummy index. A free index is an index which is not repeated in the above manner. In order to avoid confusion, the same letter should not be used to denote more than one free index in an equation. The consistency of an equation may be quickly checked by noting whether the free indices appear in the same position (superscript or subscript) throughout the equation. For instance,

$$A^i B^j = E_k F^k D^i G^j$$

is consistent, but

$$A^iB^j = D_iE^j$$

is not consistent.

A general transformation relating two coordinate systems α^i, β^i is defined by

$$\beta^i = f\,(\alpha^1, \alpha^2, \ldots, \alpha^n), \quad i = 1,2,\ldots,n \tag{2-1}$$

where f could either be an analytic or numerical transformation. In order to ensure that the transformation be a proper one-to-one invertible mapping we should be able to solve for the α^i, such that

$$\alpha^i = f^{-1}\,(\beta^1, \beta^2, \ldots, \beta^n), \quad i = 1,2,\ldots,n \tag{2-2}$$

The conditions under which f^{-1} exists are determined by the implicit function theorem (Bartle, 1975; Hildebrand, 1976). A transformation is said to be proper if the Jacobian determinant does not vanish. For a three-dimensional space J is of the form

$$J = \begin{vmatrix} \dfrac{\partial\beta^1}{\partial\alpha^1} & \dfrac{\partial\beta^1}{\partial\alpha^2} & \dfrac{\partial\beta^1}{\partial\alpha^3} \\ \dfrac{\partial\beta^2}{\partial\alpha^1} & \dfrac{\partial\beta^2}{\partial\alpha^2} & \dfrac{\partial\beta^2}{\partial\alpha^3} \\ \dfrac{\partial\beta^3}{\partial\alpha^1} & \dfrac{\partial\beta^3}{\partial\alpha^2} & \dfrac{\partial\beta^3}{\partial\alpha^3} \end{vmatrix} \tag{2-3}$$

Recall from calculus that the Jacobian represents the ratio of elemental volumes for a change of variables in a volume integral, i.e.

$$\frac{dV}{dV_o} = \frac{\partial(\beta^1, \beta^2, \ldots, \beta^n)}{\partial(\alpha^1, \alpha^2, \ldots, \alpha^n)} = J \tag{2-4}$$

where dV_o is the initial elementary volume. Should the Jacobian of the transformation vanish an elemental volume would be mapped into a point or vice versa. Clearly this violates the requirement that the transformation be a one-to-one mapping; thus we should consider only proper transformations.

II.2. Covariant and Contravariant Vectors

Consider a proper coordinate transformation of the form

$$\xi^i = \xi^i(x^1, x^2,\dots,x^n), \quad i = 1,2,\dots,n \tag{2-5}$$

Now the differential along a coordinate direction is given by

$$d\xi^i = \frac{\partial \xi^i}{\partial x^j} dx^j \tag{2-6}$$

where a summation on j is implied. A vector whose components transform in this manner is said to be a contravariant vector. The general transformation rule for contravariant vectors is given by

$$A^i = B^j \frac{\partial \xi^i}{\partial x^j} \tag{2-7}$$

Differentiating the transformation of the coordinate differential (2-6) with respect to time yields

$$\frac{d\xi^i}{dt} = \frac{\partial \xi^i}{\partial x^j} \frac{dx^j}{dt} \tag{2-8}$$

where $\frac{dx^j}{dt} = V^j$ are the components of the velocity vector. Thus, the velocity is a contravariant vector, as are the acceleration and force vectors. Consider a scalar function

$$\phi = \phi(x^1, x^2,\dots,x^n)$$

whose partial derivative with respect to a new coordinate system is

$$\frac{\partial \phi}{\partial \xi^i} = \frac{\partial \phi}{\partial x^j} \frac{\partial x^j}{\partial \xi^i} \tag{2-9}$$

A vector whose components transform in this way is said to be a covariant vector. Observe that $\frac{\partial \phi}{\partial x^j}$ are the components of the gradiant vector. The general transformation rule for covariant vectors is

$$A_i = B_j \frac{\partial x^j}{\partial \xi^i} \tag{2-10}$$

The concept of contravariant and covariant vectors can also be introduced by considering the differential of a position vector. Any line element vector $d\underset{\sim}{r}$ is related to the differential change of coordinate $d\xi^i$ by

$$d\underset{\sim}{r} = \frac{\partial \underset{\sim}{r}}{\partial \xi^i} d\xi^i \tag{2-11}$$

where $\underset{\sim}{r}$ is the position vector, and the expression is summed over the values of i. It is convenient to define the derivative $\frac{\partial \underset{\sim}{r}}{\partial \xi^i}$ as the ith covariant base vector $\underset{\sim}{a}_i$, i.e.

$$\underset{\sim}{a}_i = \frac{\partial \underset{\sim}{r}}{\partial \xi^i} \tag{2-12}$$

so that the line element vector can be written as

$$d\underset{\sim}{r} = d\xi^i \underset{\sim}{a}_i \tag{2-13}$$

More generally, any vector $\underset{\sim}{A}$ can be decomposed in the same manner as

$$\underset{\sim}{A} = A^i \underset{\sim}{a}_i \tag{2-14}$$

where A^i denotes the contravariant components of the vector $\underset{\sim}{A}$. The reciprocal (contravariant) base vectors are defined by

$$\underset{\sim}{a}^i = \frac{1}{J} (\underset{\sim}{a}_j \times \underset{\sim}{a}_k) \tag{2-15}$$

with (i,j,k) in cyclic order (e.g., if i = 1 then j = 2 and k = 3), such that

$$\underset{\sim}{a}^i \cdot \underset{\sim}{a}_j = \delta^i_j \tag{2-16}$$

where the Kronecker delta δ^i_j is a mixed second-rank tensor defined by

$$\delta^i_j = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases} \tag{2-17}$$

and J is the Jacobian defined by

$$J = \underset{\sim}{a}_i \cdot (\underset{\sim}{a}_j \times \underset{\sim}{a}_k) \tag{2-18}$$

For convenience, we shall define

$$\underset{\sim}{b}^i = \underset{\sim}{a}_j \times \underset{\sim}{a}_k = J\underset{\sim}{a}^i$$

So that the ℓth component of the ith contravariant base vector $\underset{\sim}{b}^i$ can be written as

$$b_\ell^i = \varepsilon_{\ell mn} a_j^m \, a_k^n \tag{2-19}$$

where (i,j,k) is in cyclic order and the alternating tensor $\varepsilon_{\ell mn}$ is

$$\varepsilon_{\ell mn} = \begin{cases} 1 & \text{if } \ell mn \text{ is an even permutation of } 123 \\ -1 & \text{if } \ell mn \text{ is an odd permutation of } 123 \\ 0 & \text{if any two indices are equal} \end{cases} \tag{2-20}$$

A vector $\underset{\sim}{A}$ could also be decomposed in terms of the contravariant base vectors $\underset{\sim}{a}^i$ as

$$\underset{\sim}{A} = A_i \underset{\sim}{a}^i \tag{2-21}$$

where A_i are the covariant components of the vector $\underset{\sim}{A}$. The covariant and contravariant base vectors are not usually of unit length, but a set of unit vectors tangent and normal to the coordinate lines can be defined as

$$\underset{\sim}{e}_i = \frac{\underset{\sim}{a}_i}{|\underset{\sim}{a}_i|} \tag{2-22a}$$

$$\underset{\sim}{e}^i = \frac{\underset{\sim}{a}^i}{|\underset{\sim}{a}^i|} \tag{2-22b}$$

respectively. The contravariant components of a vector, for example velocity, point along the tangent to coordinate curves. Covariant components, such as the gradiant of a scalar, point in the direction normal to coordinate surfaces. The orientation of the base vectors are shown in figure 1. The reason no distinction between contravariant and covariant vectors occurs for orthogonal coordinates can be seen by examining (2-15). When orthogonal unit base vectors are used the ith contravariant base vector defined by (2-15) will be equivalent to the ith covariant base vector.

Contravariant and covariant vectors are special cases of tensors. The components of a tensor obey the following transformation rule:

$$A^{\alpha_1\alpha_2\ldots\alpha_r}_{\beta_1\beta_2\ldots\beta_s} = \left[\frac{\partial(x^1,x^2,\ldots x^n)}{\partial(\xi^1,\xi^2,\ldots\xi^n)}\right]^N B^{a_1a_2\ldots a_r}_{b_1b_2\ldots b_s}\frac{\partial\xi^{\alpha_1}}{\partial x^{a_1}}\frac{\partial\xi^{\alpha_2}}{\partial x^{a_2}}\ldots\frac{\partial\xi^{\alpha_r}}{\partial x^{a_r}}$$

$$\cdot\frac{\partial x^{b_1}}{\partial\xi^{\beta_1}}\frac{\partial x^{b_2}}{\partial\xi^{\beta_2}}\ldots\frac{\partial x^{b_s}}{\partial\xi^{\beta_s}} \tag{2-23}$$

The exponent N of the Jacobian is called the weight of the tensor field. For $N=0$ the tensor field is absolute; if $N\neq 0$ the tensor field is relative of weight N. The tensor is contravariant of order r and covariant of order s. By setting $N=0$ and alternately $r=1$, $s=1$ the previously defined transformation rules for contravariant (2-7) and covariant (2-10) vectors are recovered.

II.3. The Metric Tensor

In mechanics, the notion of invariance is of fundamental importance. Certainly the meaning of a physical law should not be dependent upon the choice of the coordinate system; therefore, we seek to formulate the equations of fluid mechanics in an invariant form. A tensor represents an invariant quantity whose components may be transformed from one coordinate system to another using (2-23). Thus, equations expressed in curvilinear tensor form are invariant with respect to coordinate transformations.

As an example of the idea of invariance, first consider a scalar function such as temperature. A scalar is an absolute tensor of zero order. The temperature at a point on a sphere has the same value no matter if the point is referenced by a Cartesian or spherical coordinate system; a scalar is invariant. By the same token, a transformation of coordinates only changes the components of a vector; the vector itself is an invariant. A vector (contravariant components and covariant basis)

$$\underset{\sim}{A} = A^i \underset{\sim}{a}_i \tag{2-24}$$

can be transformed into an alternate coordinate system using (2-7) and (2-10) to obtain

$$\underset{\sim}{A} = A^i \underset{\sim}{a}_i = \{\bar{A}^k \frac{\partial \xi^i}{\partial x^k}\}\{\bar{\underset{\sim}{a}}_\ell \frac{\partial x^\ell}{\partial \xi^i}\} = \bar{A}^k \bar{\underset{\sim}{a}}_\ell \delta^\ell_k = \bar{A}^k \bar{\underset{\sim}{a}}_k = \bar{\underset{\sim}{A}} \tag{2-25}$$

demonstrating the invariance of the vector.

The differential arc length for Cartesian coordinates

$$(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2 \tag{2-26}$$

is another example of a scalar invariant. A Euclidean space is one in which it is possible to define such a coordinate system. For a n-dimensional Euclidean space the differential arc length is

$$(ds)^2 = \delta_{ij} dx^i dx^j \tag{2-27}$$

where the Euclidean metric tensor δ_{ij} has the values

$$\delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases} \tag{2-28}$$

Now the x^i represent Cartesian coordinates. Applying a coordinate transformation to dx^i and dx^j gives the general elemental arc length

$$(ds)^2 = \delta_{ij} (\frac{\partial x^i}{\partial \xi^k} d\xi^k)(\frac{\partial x^j}{\partial \xi^\ell} d\xi^\ell) = \delta_{ij} \frac{\partial x^i}{\partial \xi^k} \frac{\partial x^j}{\partial \xi^\ell} d\xi^k d\xi^\ell \tag{2-29a}$$

$$(ds)^2 = g_{\ell k}\, d\xi^k d\xi^\ell \qquad (2\text{-}29b)$$

where

$$g_{\ell k} = \delta_{ij} \frac{\partial x^i}{\partial \xi^k} \frac{\partial x^j}{\partial \xi^\ell} \qquad (2\text{-}29c)$$

defines the metric tensor. Observe that the metric tensor is symmetric, i.e. $g_{ij} = g_{ji}$. Note that one need not always define the metric tensor as a transformation between Cartesian and general coordinates. If the metric tensor is known in one coordinate system the metric tensor in any other coordinate system can be found using the tensor transformation rule (2-23):

$$\bar{g}_{ij} = g_{\ell k} \frac{\partial \eta^\ell}{\partial \xi^i} \frac{\partial \eta^k}{\partial \xi^j}$$

where $g_{\ell k}$ is the metric tensor for the general η^ℓ coordinates. The metric tensor can also be defined by considering the scalar product of two line elements (2-13)

$$(ds)^2 = d\underset{\sim}{r} \cdot d\underset{\sim}{r} = \underset{\sim}{a}_i \cdot \underset{\sim}{a}_j\, d\xi^i\, d\xi^j$$

where

$$g_{ij} = \underset{\sim}{a}_i \cdot \underset{\sim}{a}_j$$

defines the metric tensor. Since the metric tensor is defined in the same way in every coordinate system it is an isotropic tensor. In fact, the metric tensor is the most general second-order isotropic tensor.

An inverse (contravariant) metric tensor g^{ij} is defined by

$$g^{ij} g_{jk} = \delta^i_k \qquad (2\text{-}30)$$

Notice that the determinant of the g_{ij} matrix is the square of the Jacobian,

$$g = |g_{ij}| = J^2 \tag{2-31}$$

and by using (2-30) we have

$$g^{-1} = |g^{ij}| = J^{-2} \tag{2-32}$$

The cofactor of the g_{ij} matrix is defined as

$$\frac{1}{2}\,\varepsilon^{ik\ell}\,\varepsilon^{jmn}\,g_{km}g_{\ell n} \tag{2-33}$$

from which the inverse metric tensor components can be found:

$$g^{ij} = \frac{1}{2g}\,\varepsilon^{ik\ell}\varepsilon^{jmn}\,g_{km}g_{\ell n} \tag{2-34a}$$

$$= \frac{1}{g}\,(g_{km}g_{\ell n} - g_{kn}g_{\ell m}) \tag{2-34b}$$

where the permutation symbol $\varepsilon^{\ell mn}$ is defined in the same way as $\varepsilon_{\ell mn}$ and in (2-34b) (i,k,ℓ), (j,m,n) are in cyclic order.

The metric tensor takes a simpler form when the coordinate system is orthogonal:

$$g_{ij} = \begin{cases} 0 & \text{for } i \neq j \\ h_i^2 & \text{for } i = j \end{cases} \tag{2-35}$$

where h_i is the metric scale factor. The inverse metric tensor is then simply

$$g^{ij} = \begin{cases} 0 & \text{for } i \neq j \\ \dfrac{1}{h_i^2} & \text{for } i = j \end{cases} \tag{2-36}$$

The line element (2-29b) for an orthogonal coordinate system is

$$(ds)^2 = (h_1 d\xi^1)^2 + (h_2 d\xi^2)^2 + (h_3 d\xi^3)^2 \tag{2-37}$$

For an axisymmetric cylindrical coordinate system, $(\xi^1,\xi^2,\xi^3) = (x,r,\theta)$, the elements of the metric tensor are

$$g_{11} = h_1^2 = 1$$
$$g_{22} = h_2^2 = 1 \qquad (2\text{-}38)$$
$$g_{33} = h_3^2 = r^2$$

For two dimensional streamline-normal coordinates, $(\xi^1, \xi^2) = (s, n)$, the nonzero components of the metric tensor are

$$g_{11} = h_1^2 = \left(1 + \frac{n}{R_o}\right)^2 \qquad (2\text{-}39)$$
$$g_{22} = h_2^2 = 1$$

where R_o is the radius of curvature of a reference streamline or body surface.

The length of a vector is given by

$$|\underset{\sim}{A}|^2 = \underset{\sim}{A} \cdot \underset{\sim}{A} = g_{ij} A^i A^j \qquad (2\text{-}40)$$

from which the associated tensor of A^j is defined as

$$A_i = g_{ij} A^j \qquad (2\text{-}41)$$

and by use of (2-30)

$$A^j = g^{ij} A_i \qquad (2\text{-}42)$$

This provides a convenient way to raise or lower indices. Thus, the scalar product of two vectors can be expressed as

$$\underset{\sim}{A} \cdot \underset{\sim}{B} = g_{ij} A^i B^j = A_j B^j \qquad (2\text{-}43)$$

From the definition of a scalar product, the angle between two vectors is

$$\cos\theta = \frac{g_{ij}A^iB^j}{|\underset{\sim}{A}|\ |\underset{\sim}{B}|} \tag{2-44}$$

and the angle between two coordinate lines is given by

$$\cos\theta_{ij} = \frac{g_{ij}}{\sqrt{g_{ii}g_{jj}}} \tag{2-45}$$

II.4. Physical Components of Tensors

A particularly inconvenient property of curvilinear tensors is that the dimensions carried by the tensor components do not necessarily correspond to the physically meaningful dimensions of the quantity represented by the tensor. For example, in Cartesian coordinates each coordinate direction has the dimension of length. In contrast, for cylindrical coordinates not every coordinate direction has the dimension of length. Consider the contravariant velocity components in cylindrical coordinates $(\xi^1,\xi^2,\xi^3) = (x,r,\theta)$:

$$V^i = \frac{d\xi^i}{dt} \tag{2-46}$$

It is evident that V^3 does not possess the usual [L/T] dimensions of velocity; instead, its dimension is [1/T]. Therefore, tensor components may not represent physically measurable quantities.

Methods by which the tensor components may be converted to physical components are neatly presented by Truesdell (1953). The physical component can be defined as the magnitude of the ith component projected onto the ith coordinate direction:

$$\begin{aligned} A(i) &= (A^i\underset{\sim}{a}_i) \cdot \underset{\sim}{e}_i \qquad \text{(no sum on i)} \\ &= \sqrt{g_{ii}}\, A^i\underset{\sim}{e}_i \cdot \underset{\sim}{e}_i \\ &= \sqrt{g_{ii}}\, A^i \end{aligned}$$

where $\underset{\sim}{e}_i$ is a unit covariant base vector. Based on this definition the physical components of a covariant or contravariant vector are given by

$$A(i) = \sqrt{g_{ii}}\ A^i = \sqrt{g_{ii}}\ g^{ij}\ A_j \text{ (no sum on i)} \tag{2-47}$$

The physical components of higher-order tensors may be obtained by observing how they are defined in terms of lower-order tensors. If

$$C^i = A^i_j B^j \tag{2-48}$$

then an equally valid equation for the physical components is

$$C(i) = A(ij)\ B(j) \tag{2-49}$$

Using (2-47) in (2-48) yields

$$\frac{C(i)}{\sqrt{g_{ii}}} = A^i_j \frac{B(j)}{\sqrt{g_{jj}}} \tag{2-50}$$

which leads to

$$A(ij) = \sqrt{\frac{g_{ii}}{g_{jj}}}\ A^i_j \tag{2-51}$$

Further examples and discussion are given by Truesdell.

II.5. Derivatives of Tensors: Covariant Differentiation

In order to express quantities which involve partial derivatives, for instance the Laplacian, a partial derivative operation which is valid in curvilinear coordinates is required. The need for a general partial derivative is demonstrated by observing that

$$\frac{\partial \underset{\sim}{A}}{\partial \xi^j} = \frac{\partial A^i}{\partial \xi^j} \underset{\sim}{a}_i + A^i \frac{\partial \underset{\sim}{a}_i}{\partial \xi^j} \tag{2-52}$$

Immediately it is clear that the variation of the base vectors along coordinate lines will make partial differentiation more complicated in curvilinear coordinates. In Cartesian coordinates, the partial derivatives always give higher order tensors. This is not true for curvilinear coordinates. For

example, although $\frac{\partial \phi}{\partial \xi^i}$ is a covariant vector for any scalar function ϕ, the second derivative $\frac{\partial^2 \phi}{\partial \xi^i \partial \xi^j}$ does not give a covariant second-order tensor. In this respect, the curvilinear system is not as convenient as the Cartesian system. It is important to define a derivative which preserves tensor character under coordinate transformations so that the effect of coordinate transformations need not be considered explicitly. A derivative with the desired tensor transformation properties can be defined as follows:

$$A^{\alpha_1\alpha_2\cdots\alpha_r}_{\beta_1\beta_2\cdots\beta_s,m} = \frac{\partial}{\partial \xi^m} A^{\alpha_1\alpha_2\cdots\alpha_r}_{\beta_1\beta_2\cdots\beta_s} + \sum_{i=1}^{r} \Gamma^{\alpha_i}_{\mu_i m} A^{\alpha_1\cdots\mu_i\cdots\alpha_r}_{\beta_1\beta_2\cdots\beta_s} - \sum_{j=1}^{s} \Gamma^{\mu_j}_{\beta_j m} A^{\alpha_1\alpha_2\cdots\alpha_r}_{\beta_1\beta_2\cdots\mu_j\cdots\beta_s} \tag{2-53}$$

This is called the covariant derivative of the general tensor $A^{\alpha_1\alpha_2\cdots\alpha_r}_{\beta_1\beta_2\cdots\beta_s}$ and Γ^i_{jk} is the Christoffel symbol defined by

$$\Gamma^i_{jk} = \frac{1}{2} g^{i\ell}\left\{\frac{\partial g_{\ell j}}{\partial \xi^k} + \frac{\partial g_{k\ell}}{\partial \xi^j} - \frac{\partial g_{jk}}{\partial \xi^\ell}\right\} \tag{2-54}$$

It is clear that the Christoffel symbol is symmetric in j and k. Furthermore, Γ^i_{ij} is related to the Jacobian J or $\sqrt{g}$ by

$$\Gamma^i_{ij} = \Gamma^i_{ji} = \frac{1}{2} \frac{\partial(\ell n\ g)}{\partial \xi^i} = \frac{1}{2g} \frac{\partial g}{\partial \xi^i} \tag{2-55a}$$

or

$$\Gamma^i_{ij} = \Gamma^i_{ji} = \frac{1}{J} \frac{\partial J}{\partial \xi^i} \tag{2-55b}$$

since $g = J^2$.

For orthogonal curvilinear coordinates the Christoffel symbols are related to the metric scale factors as follows:

$$\begin{aligned} \Gamma^i_{jk} &= 0 & \Gamma^i_{ij} &= \Gamma^i_{ji} = \frac{1}{h_i} \frac{\partial h_i}{\partial \xi^j} \\ \Gamma^i_{ii} &= \frac{1}{h_i} \frac{\partial h_i}{\partial \xi^i} & \Gamma^i_{jj} &= \frac{-h_j}{h_i^2} \frac{\partial h_j}{\partial \xi^i} \end{aligned} \tag{2-56}$$

where i,j,k are in cyclic order.

It is noted that both $\frac{\partial}{\partial \xi^m} A^{\alpha_1 \alpha_2 \cdots \alpha_r}_{\beta_1 \beta_2 \cdots \beta_s}$ and Γ^i_{jk} are not tensors; however, the covariant derivative (2-53) does preserve the tensor transformation rule under coordinate transformation and is therefore a higher order tensor. The result of equation (2-53) is that we need not consider explicitly the effect of coordinate transformations on tensor derivatives. For example, since $g_{ij,k}$ and $g^{ij}_{,k}$ are zero in Cartesian coordinates, they are zero in all coordinate systems.

From equation (2-53), the covariant derivative of a covariant vector $\underset{\sim}{A}$ with respect to ξ^i is given by

$$A_{i,j} = \frac{\partial A_i}{\partial \xi^j} - \Gamma^k_{ij} A_k \tag{2-57}$$

Similarly, the covariant derivative of a contravariant vector A^i is given by

$$A^i_{,j} = \frac{\partial A^i}{\partial \xi^j} + \Gamma^i_{jk} A^k \tag{2-58}$$

The covariant derivative of the physical components of a contravariant vector may be found by introducing the physical components into (2-58)

$$A^i_{,m} = g_{ii}^{-1/2} \, g_{mm}^{1/2} \, \{\frac{\partial A(i)}{\partial \xi(m)} + D(im)\, A(i) + \Gamma\, (imn)\, A(n)\} \tag{2-59}$$

where

$$\frac{\partial}{\partial \xi(m)} = g_{mm}^{-1/2} \frac{\partial}{\partial \xi^m} \tag{2-59a}$$

$$D(im) = g_{ii}^{1/2} \frac{\partial g_{ii}^{-1/2}}{\partial \xi(m)} = - \frac{1}{2g_{ii}} \frac{\partial g_{ii}}{\partial \xi(m)} \tag{2-59b}$$

and $\Gamma(imn)$ is the physical component of the Christoffel symbol defined by

$$\Gamma(imn) = g_{ii}^{1/2} \, g_{mm}^{-1/2} g_{nn}^{-1/2} \, \Gamma^i_{mn} \tag{2-59c}$$

If $A^i = A^i\,(\underset{\sim}{\xi}(t),t)$, then the intrinsic (total) derivative of A^i with respect to time is

$$\frac{\delta A^i}{\delta t} = \frac{\partial A^i}{\partial t} + A^i_{,j} \frac{d\xi^j}{dt} \qquad (2\text{-}60)$$

Since $\frac{d\xi^j}{dt}$ is the velocity, this is an expression for the material derivative of $\underset{\sim}{A}$.

With the foregoing tensor concepts, we are now in a position to recast many of the familiar operations of vector analysis into curvilinear tensor form. Although these are not directly employed in subsequent sections, they are summarized in the Appendix since they are useful in gaining insights into the equations of motion and relating the tensor equations to the more familiar vector forms.

III. FUNDAMENTAL EQUATIONS OF FLUID MECHANICS

The equations governing fluid motion in a nonorthogonal coordinate system can be derived using the methods of curvilinear tensor analysis. The results of the previous section provide the necessary framework for writing these equations in an invariant form; the physical meaning of the equations will be consistent under any coordinate transformation. Alternatively, the equations could be derived by the methods of vector analysis. The details of the vector analysis approach are presented by Stern and Yoo (1986).

The majority of equations are given using tensor components (contravariant or covariant) which do not necessarily carry the proper physical dimensions of the quantity they represent. In a few example cases the equations are written substituting the physical components for the tensor components. As a rule, tensor components are more easily manipulated and we prefer their use over physical components when in the process of deriving general results.

Many of the equations given here are straightforward generalizations of their Cartesian tensor counterparts. The reader is encouraged to compare the equations presented here to those in Cartesian tensor form (see, for example, Currie, 1974). Loosely speaking, the Cartesian tensor form of the equations can be generalized by replacing partial derivatives with covariant derivatives, making the velocity a contravariant vector, and keeping the proper overall covariant and/or contravariant order.

The material derivative represents the rate of change of a quantity as seen by an observer moving with a fluid particle. Therefore, the material derivative of a general tensor, which may be a function of both space and time, is given by its intrinsic (total) derivative

$$\frac{DA}{Dt} = \frac{\partial A}{\partial t} + A_{,j}\frac{d\xi^j}{dt} \qquad (3\text{-}1)$$

Now $\frac{d\xi^j}{dt}$ represents the jth contravariant velocity component. Letting $V^j = \frac{d\xi^j}{dt}$, the expression for the material derivative becomes

$$\frac{DA}{Dt} = \frac{\partial A}{\partial t} + A_{,j}V^j \qquad (3\text{-}2)$$

Notice that this expression is quite similar to the material derivative defined using Cartesian tensors, but now the usual partial derivative has been replaced by the covariant derivative. If the general tensor A represents a velocity, equation (3-1) may be written in terms of contravariant velocity components as

$$\frac{DV^i}{Dt} = \frac{\partial V^i}{\partial t} + V^j V^i_{,j} \qquad (3\text{-}3a)$$

$$= \frac{\partial V^i}{\partial t} + V^j \left\{\frac{\partial V^i}{\partial \xi^j} + \Gamma^i_{jk} V^k\right\} \qquad (3\text{-}3b)$$

This represents the fluid acceleration in general tensor form.

The material derivative of a volume integral is given by the Reynolds Transport Theorem. The theorem is useful in deriving Eulerian conservation laws from their Lagrangian equivalents. Generalizing the Cartesian tensor form of the theorem yields

$$\frac{D}{Dt}\int_{\mathbb{V}} A \, d\mathbb{V} = \int_{\mathbb{V}} \left\{\frac{DA}{Dt} + AV^i_{,i}\right\} d\mathbb{V} \qquad (3\text{-}4)$$

where A is a general tensor and $\mathbb{V}$ represents a material volume. An alternate and perhaps more familiar arrangement is

$$\frac{D}{Dt}\int_{\mathbb{V}} A \, d\mathbb{V} = \int_{\mathbb{V}} \left\{\frac{\partial A}{\partial t} + (AV^i)_{,i}\right\} d\mathbb{V} \qquad (3\text{-}5)$$

III.1. Equation of Continuity

The conservation of mass or continuity equation may be derived by letting the general tensor A represent the fluid density ρ in (3-4), by which

$$\int_{\mathbb{V}} \left[\frac{D\rho}{Dt} + \rho V^i_{,i}\right] d\mathbb{V} = 0 \qquad (3\text{-}6)$$

Since the volume considered is arbitrary the integrand must be equal to zero:

$$\frac{D\rho}{Dt} + \rho V^i_{,i} = 0 \qquad (3\text{-}7)$$

If the fluid is incompressible the volume of a material element must remain constant, in other words

$$\frac{D\rho}{Dt} = 0 \qquad (3\text{-}8)$$

Hence, the statement of mass conservation for an incompressible fluid becomes

$$V^i_{,i} = 0 \qquad (3\text{-}9)$$

which is $\nabla \cdot \underset{\sim}{V} = 0$ in vector notation.

A useful corollary to the Reynolds Transport Theorem, derived using the conservation of mass statement, is

$$\frac{D}{Dt} \int_{\forall} \rho A \, d\forall = \int_{\forall} \rho \frac{DA}{Dt} \, d\forall \qquad (3\text{-}10)$$

Substitution of the physical velocity components for the contravariant components in (3-9) gives

$$g_{ii}^{-1/2} \frac{\partial V(i)}{\partial \xi^i} + V(i) \frac{\partial g_{ii}^{-1/2}}{\partial \xi^i} + \Gamma^i_{ik} \, g_{kk}^{-1/2} \, V(k) = 0 \qquad (3\text{-}11)$$

Note the sum over both i and k. Using (2-55b)

$$\Gamma^i_{ik} = g^{-1/2} \frac{\partial g^{1/2}}{\partial \xi^k}$$

and the continuity equation may be written in the form

$$\frac{1}{g^{1/2}} \frac{\partial}{\partial \xi^i} \{ \frac{g^{1/2} V(i)}{g_{ii}^{1/2}} \} = 0 \qquad (3\text{-}12)$$

III.2. Momentum (Navier-Stokes) Equations

The conservation of linear momentum (Newton's Second Law) for an arbitrary material volume can be expressed as

$$\frac{D}{Dt}\int_{\mathcal{V}} \rho \; \underset{\sim}{V} \cdot \underset{\sim}{\ell} \, d\mathcal{V} = \int_{\mathcal{V}} \rho \underset{\sim}{F} \cdot \underset{\sim}{\ell} \, d\mathcal{V} + \int_{S} \underset{\sim}{t} \cdot \underset{\sim}{\ell} \, dS \qquad (3\text{-}13)$$

where $\underset{\sim}{V}$, $\underset{\sim}{F}$, and $\underset{\sim}{t}$ are, respectively, the velocity, body force, and surface stress vectors. Because the absolute direction of the base vectors is, in general, variable, each force should be resolved along a constant direction $\underset{\sim}{\ell}$. The need for this refinement can be seen by considering the forces on a fluid element defined by a cylindrical coordinate system.

The nature of the surface forces can be deduced by examining the forces acting on an elemental volume of fluid. If the volume in (3-13) is decreased to a point while preserving its shape, we find that the surface forces are in a state of local equilibrium:

$$\lim_{\mathcal{V}\to 0} \frac{1}{r^2} \int_S \underset{\sim}{t} \cdot \underset{\sim}{\ell} \, dS = 0 \qquad (3\text{-}14)$$

where r is a characteristic dimension of the volume. We need to find how the stress on an arbitrary surface is related to the stresses on the coordinate surfaces. Figure 1 shows a typical volume element and the stress vectors acting on each surface. Since the forces are in local equilibrium we have

$$\underset{\sim}{t}_n \, dS = \underset{\sim}{t}_1 \, dS_1 + \underset{\sim}{t}_2 \, dS_2 + \underset{\sim}{t}_3 \, dS_3 \qquad (3\text{-}15)$$

where dS refers to the area of the arbitrary surface with normal $\underset{\sim}{n}$. Note that the normal vector is composed of

$$\underset{\sim}{n} = n^i \underset{\sim}{a}_i = n_i \underset{\sim}{a}^i \qquad (3\text{-}16)$$

The outward normal vector (not necessarily a unit vector) for each coordinate surface is $\underset{\sim}{a}^i$. The surface area of each face of the element is related to dS by

$$dS_i = \underset{\sim}{a}_i \cdot \underset{\sim}{n} \, dS = \underset{\sim}{a}_i \cdot (n_j \underset{\sim}{a}^j) \, dS = n_i \, dS \qquad (3\text{-}17)$$

Substituting for each dS_i in (3-15) yields

$$\underset{\sim}{t}_n = \underset{\sim}{t}_1 n_1 + \underset{\sim}{t}_2 n_2 + \underset{\sim}{t}_3 n_3$$

The stress vector for each coordinate surface is then

$$\underset{\sim}{t}_j = \sigma^{ij} \underset{\sim}{a}_i \tag{3-18}$$

where σ^{ij} defines the stress tensor which represents the ith component of $\underset{\sim}{t}_j$
Now $\underset{\sim}{t}_n$ can be written as

$$\underset{\sim}{t}_n = (\sigma^{ij} n_j)\, \underset{\sim}{a}_i \tag{3-19}$$

Evidently, the ith contravariant component of stress on the surface with outward normal $\underset{\sim}{n}$ is

$$t^i = \sigma^{ij} n_j \tag{3-20}$$

The stress tensor components are shown in figure 2. Observe that σ^{ii} does not necessarily represent a normal stress; it is simply the stress in the ith coordinate direction. *Only in an orthogonal* coordinate system will σ^{ii} be a normal stress. Since we are considering fluids which cannot have internal stress couples (moments), angular momentum can be conserved only if the stress tensor is symmetric, i.e.

$$\sigma^{ij} = \sigma^{ji}$$

This result can be derived by considering the moment of momentum equation.

Utilizing the expression for the stress on an arbitrary surface (3-20), the momentum equation may be rewritten as

$$\int_{V} \rho \frac{DV^i}{Dt} \ell_i \, dV = \int_{V} \{\rho F^i \ell_i + \sigma^{ij}_{\ ,j} \ell_i\} \, dV \tag{3-21}$$

where (3-10) and (A-16) have been used. Since the direction $\underset{\sim}{\ell}$ is constant ($\ell_{i,k} = 0$) and both the volume and $\underset{\sim}{\ell}$ are completely arbitrary, the equality can be satisfied only if

$$\rho \frac{DV^i}{Dt} = \sigma^{ij}_{,j} + \rho F^i \qquad (3\text{-}22)$$

This result is known as Cauchy's equation of motion and it is valid for any continuum. In order to solve this equation a relation between the stress tensor and the velocity field is required.

Although a complete development of the constitutive equation for a Newtonian fluid is beyond the scope of this report, a few details of the derivation will be shown in order to make the origin and form of the final result clear. A rigorous derivation, using Cartesian tensors, is available in many texts (e.g. Batchelor (1967)).

In a motionless fluid the force on a fluid element is due solely to the internal pressure. For an incompressible fluid, this is the mechanical pressure. If the fluid is compressible, the pressure would be the thermodynamic pressure specified by an equation of state. For a static fluid the stress tensor is isotropic and can be written as

$$\sigma^{ij} = - p\, g^{ij} \qquad (3\text{-}23)$$

From this the stress on an elemental area can be determined using (3-19):

$$\underset{\sim}{t}_n = (-pg^{ij}n_j)\, \underset{\sim}{a}_i = (-pn^i)\, \underset{\sim}{a}_i = - p\underset{\sim}{n}$$

Consequently, the pressure acts normal to the surface.

For a moving fluid, a more general form of the stress tensor should be considered:

$$\sigma^{ij} = - pg^{ij} + \tau^{ij} \qquad (3\text{-}24)$$

The additional stress due solely to the motion of the fluid is the deviatoric stress τ^{ij}. The deviatoric stress is assumed to be proportional to the fluid deformation rate

$$\tau^{ij} \propto V^k_{,j}$$

and $V^k_{,j}$ can be decomposed into its symmetric and antisymmetric parts

$$V^k_{,i} = g^{jk} (e_{ij} + \Omega_{ij}) \qquad (3\text{-}25)$$

where $e_{ij} = \frac{1}{2} (V_{i,j} + V_{j,i})$ and $\Omega_{ij} = \frac{1}{2} (V_{i,j} - V_{j,i})$. Since the stress tensor must be symmetric the deviatoric stress cannot depend upon the antisymmetric portion of the deformation rate. Assuming that a linear relation between the deviatoric stress and strain rate (e_{ij}) exists,

$$\tau^{ij} = D^{ij\ell m} e_{\ell m} \qquad (3\text{-}26)$$

where $D^{ij\ell m}$ is a fourth-order isotropic tensor (in fact, it is a composition of $g^{\ell m}$). Hence, τ^{ij} is related to $e_{\ell m}$ by a linear combination of isotropic, fourth-order tensors:

$$\tau^{ij} = \lambda g^{ij} g^{\ell m} e_{\ell m} + \mu (g^{mi} g^{\ell j} + g^{i\ell} g^{jm}) e_{\ell m} \qquad (3\text{-}27)$$

where μ is the dynamic viscosity and λ is the second viscosity coefficient. This can be simplified to

$$\tau^{ij} = \lambda g^{ij} V^i_{,i} + 2\mu e^{ij} \qquad (3\text{-}28)$$

where $e^{ij} = g^{i\ell} g^{jm} e_{\ell m}$. Thus, for an incompressible fluid ($V^i_{,i}=0$) the Newtonian stress tensor is

$$\sigma^{ij} = - g^{ij} p + 2\mu e^{ij} = -g^{ij} p + \mu g^{ik} g^{j\ell} (V_{k,\ell} + V_{\ell,k}) \qquad (3\text{-}29)$$

In (3-22) the divergence of the stress tensor is required. Carrying out this operation, using the fact that $g^{ij}_{,\ell} = 0$, gives

$$\sigma^{ij}_{,j} = - g^{ij} p_{,j} + \mu (g^{j\ell} V^i_{,\ell j} + g^{ik} V^j_{,kj}) \qquad (3\text{-}30)$$

For most fluid mechanics applications it can be assumed that the space is Euclidean. This guarantees that $V^j_{,kj} = V^j_{,jk}$, which, by the equation of continuity, is equal to zero. Then the divergence of the stress tensor is

$$\sigma^{ij}_{,j} = - g^{ij} p_{,j} + \mu g^{j\ell} V^{i}_{,\ell j} \qquad (3\text{-}31)$$

Notice that the only free index is i, which makes the result a contravariant vector.

The Navier-Stokes equations for an incompressible fluid are obtained by substituting (3-31) into the equation of motion (3-22). The result is

$$\frac{\partial V^{i}}{\partial t} + V^{m} V^{i}_{,m} = - g^{im} \frac{p_{,m}}{\rho} + \frac{\mu}{\rho} g^{mn} V^{i}_{,mn} + F^{i} \qquad (3\text{-}32)$$

Replacing V^{i} by $g_{ii}^{-1/2}$ V(i) and making use of (2-59) and (A-15) yields the following Navier-Stokes equations with physical components as dependent variables:

$$\frac{\partial V(i)}{\partial t} + V(m) \frac{\partial V(i)}{\partial \xi(m)} + D(im)\, V(m)\, V(i) + \Gamma(imn)\, V(m)\, V(n) + g(im) \frac{\partial}{\partial \xi(m)} \left(\frac{p}{\rho}\right)$$

$$-\nu\left\{\nabla^{2} V(i) + 2E(im) \frac{\partial V(i)}{\partial \xi(m)} + 2C(imn) \frac{\partial V(n)}{\partial \xi(m)} + P(i)\, V(i) + Q(in)\, V(n)\right\} - F(i) = 0$$

$$(3\text{-}33)$$

where

$$F(i) = g_{ii}^{1/2} F^{i} \qquad (3\text{-}34)$$

and E(im), C(imn), P(i) and Q(in) are defined in equations (A-15a) through (A-15d). The Laplacian of V(i) is given by equation (A-14) as

$$\nabla^{2} V(i) = (g_{mm} g_{nn})^{-1/2} g(mn) \frac{\partial^{2} V(i)}{\partial \xi^{m} \partial \xi^{n}} + f(m) \frac{\partial V(i)}{\partial \xi(m)} \qquad (3\text{-}35)$$

These equations are rather cumbersome when expanded for a given value of i because many terms include implied summations. Even though both g(ij) and Γ(ijk) are symmetric they have 6 and 18 independent components, respectively. Note that using the physical velocity components has introduced additional terms like D(im)V(m)V(i) into the equation which are not present in the contravariant form (3-32).

III.3. Energy Equation

By applying the first law of thermodynamics to a fluid volume, an equation expressing the conservation of energy can be obtained. Since energy is a scalar, the formulation of the *energy equation* provides a general example of how transport equations for other scalar quantities may be derived. The first law states that for a given system the rate of change of total energy is equal to the rate of work done plus the rate of heat added. For an arbitrary fluid volume we have

$$\frac{D}{Dt}\int_{\forall} \rho \left(e + \frac{1}{2}\, \underset{\sim}{V} \cdot \underset{\sim}{V}\right) d\forall = \int_S \underset{\sim}{V} \cdot \underset{\sim}{t}\, dS + \int_{\forall} \underset{\sim}{V} \cdot \rho \underset{\sim}{F}\, d\forall - \int_S \underset{\sim}{q} \cdot \underset{\sim}{n}\, dS \tag{3-36}$$

where e is the internal energy per unit mass and $\underset{\sim}{q}$ denotes the conductive heat transfer (positive for heat leaving the volume). The left hand side represents the rate of change of total (internal and kinetic) energy and the right hand side is composed of the rate of work done by surface forces, rate of work done by body forces, and the rate of heat addition respectively. Converting (3-36) to general tensor form and using (3-10) yields

$$\int_{\forall} \rho \frac{D}{Dt} \left(e + \frac{1}{2}\, g_{ij} V^i V^j\right) d\forall = \int_S g_{ij} t^i V^j\, dS + \int_{\forall} \rho g_{ij} V^j F^i d\forall - \int_S g_{ij} q^i n^j\, dS \tag{3-37}$$

After substituting (3-20) for the surface force vector and using the divergence theorem on the surface integrals, we obtain

$$\int_{\forall} \rho \frac{D}{Dt} \left(e + \frac{1}{2}\, g_{ij} V^i V^j\right) d\forall = \int_{\forall} [(\sigma^{ij} V_i)_{,j} + \rho\, V_i F^i - q^i_{,i}\,] d\forall \tag{3-38}$$

Since the volume considered is completely arbitrary the equality holds only if

$$\rho \frac{D}{Dt} \left(e + \frac{1}{2}\, g_{ij} V^i V^j\right) = (\sigma^{ij} V_i)_{,j} + \rho V_i F^i - q^i_{,i} \tag{3-39}$$

The kinetic energy term may be rewritten as follows:

$$\frac{D}{Dt}\left(\frac{1}{2} g_{ij} V^i V^j\right) = \frac{1}{2} g_{ij} V^i \frac{DV^j}{Dt} + \frac{1}{2} g_{ij} V^j \frac{DV^i}{Dt}$$
$$= g_{ij} V^i \frac{DV^j}{Dt}$$
$$= V_i \frac{DV^j}{Dt}$$

The mechanical energy equation (Cauchy's equation (3-22) multiplied by V_i) may now be identified and subtracted out from (3-39), leaving

$$\rho \frac{De}{Dt} = \sigma^{ij} V_{i,j} - q^i_{,i} \qquad (3\text{-}40)$$

which is a transport equation for the internal energy. The heat flux vector can be related to the temperature field through Fourier's law:

$$q^i = - g^{ij} K T_{,j} \qquad (3\text{-}41)$$

where K is the thermal conductivity. Substituting for the heat flux in (3-40) gives

$$\rho \frac{De}{Dt} = (g^{ij} K T_{,j})_{,i} + \sigma^{ij} V_{i,j} \qquad (3\text{-}42)$$

The last term of the above equation represents the increase in internal energy due to the work done in deforming a fluid element; in other words, conversion of mechanical energy to thermal energy by the action of surface forces. Replacing the stress tensor with its incompressible fluid form (3-29), we have

$$\sigma^{ij} V_{i,j} = (-pg^{ij} + 2\mu e^{ij})\, V_{i,j}$$
$$= -pV^j_{,j} + 2\mu e^{ij} V_{i,j} \qquad (3\text{-}43)$$
$$= 2\mu e^{ij}\, V_{i,j}$$

Further insight can be gained by splitting $V_{i,j}$ into its symmetric and anti-symmetric parts:

$$V_{i,j} = \frac{1}{2}(V_{i,j} + V_{j,i}) + \frac{1}{2}(V_{i,j} - V_{j,i})$$

$$= e_{ij} + \Omega_{ij}$$

Now substituting this result into (3-43) gives

$$\sigma^{ij}V_{i,j} = 2\mu e^{ij}(e_{ij} + \Omega_{ij})$$

and recognizing that the multiplication of a symmetric tensor and anti-symmetric tensor is zero, we obtain

$$\Phi = \sigma^{ij}V_{i,j} = 2\mu e^{ij}e_{ij}$$

$$= \mu(V^i_{,j}\, V^j_{,i} + g^{\ell m}g_{ij}V^i_{,\ell}V^j_{,m}) \tag{3-44}$$

where Φ denotes the dissipation function. The dissipation function is a positive definite quantity which represents the irreversible conversion of mechanical energy into thermal energy due to the action of viscosity. The energy equation for an incompressible fluid may now be written as

$$\rho \frac{De}{Dt} = (g^{ij}KT_{,j})_{,i} + \Phi \tag{3-45}$$

III.4. Caution on Approximations

At this point it seems appropriate to make some general comments regarding the invariance of the governing equations. Recall that a tensor represents a quantity which is invariant under coordinate transformations. The equations presented above are written in general tensor form and therefore, are invariant. However, when the covariant derivatives are expanded and physical components are introduced, some parts of the equations will not be tensors. For example, $\Gamma^i_{jk} A^k$ is a part of the covariant derivative, but it is not a tensor. The covariant derivative itself is a tensor and thus represents an invariant quantity. It is also important to note that after expansion some individual terms of an equation may appear to be identical, but in reality they may have originated from different invariant groups that do not have the same physical meaning.

Another important point to be noted is that when approximations are made to an initially invariant equation the resulting equation may lose the invariance property. A familiar example of this is provided by the approximation of the Navier-Stokes equations to form the boundary-layer equations. Since the boundary-layer approximation neglects only certain terms in the original invariant groups, the resulting equations are not independent of the coordinate system. The implications of this fact were discussed some time ago by Kaplun (1954).

Another problem inherent in making approximations based on the order of magnitude of individual terms is that of estimating the contribution from the geometric coefficients. Clearly, one should consider for each term the order of magnitude of both the independent and dependent variables. Even though a quantity composed of dependent variables may be of small order, it may in turn be multiplied by a large geometric coefficient, such that the net result is a term of first order significance.

IV. EQUATIONS FOR TURBULENT FLOW

It is generally assumed that the equations derived in the previous section apply in a turbulent flow at all instants of time. For most practical purposes, however, we follow Reynolds and decompose the instantaneous values of flow variables, such as velocity and pressure, into a mean value and a fluctuation about that mean. Henceforth, the instantaneous velocity V^i is decomposed into $V^i + v^i$ and the instantaneous pressure is decomposed into $p + p'$. The equations governing the mean flow are then derived from those given in the previous section by making this substitution and taking the mean values.

IV.1. Continuity Equation

Introducing the mean and fluctuating components into the continuity equation (3-9) gives

$$V^i_{,i} + v^i_{,i} = 0 \tag{4-1}$$

and averaging the equation we have

$$V^i_{,i} = 0 \tag{4-2}$$

From (3-12), this may be written in conservative form as

$$g^{-1/2} \frac{\partial}{\partial \xi^i} \{g^{1/2} g_{ii}^{-1/2} V(i)\} = 0 \tag{4-3}$$

or, in non-conservative form, as

$$\frac{\partial V(i)}{\partial \xi(i)} + [D(ii) + \Gamma(nni)]\, V(i) = 0 \tag{4-4}$$

Thus, the mean velocity field satisfies the usual continuity equation (3-9) or (3-12). The fluctuating velocity field also satisfies the continuity equation since subtracting (4-2) from (4-1) yields

$$v^i_{,i} = 0 \tag{4-5}$$

IV.2. Momentum (Reynolds) Equations

Substituting the mean and fluctuating components of velocity and pressure into the Navier-Stokes equations (3-32) and Reynolds averaging the result gives

$$\frac{\partial V^i}{\partial t} + V^m V^i_{,m} = -\frac{1}{\rho} g^{im} P_{,m} + \nu g^{mn} V^i_{,mn} - (\overline{v^i v^m})_{,m} + F^i \tag{4-6}$$

These are the momentum equations, or Reynolds equations, for turbulent flow. These equations are identical to the original Navier-Stokes equations except for the appearance of the Reynolds stress tensor $\overline{v^i v^m}$. The additional Reynolds stress terms arise as a result of the nonlinearity of the equation of motion, and can be expressed in terms of their physical components as

$$(\overline{v^i v^m})_{,m} = g_{ii}^{-1/2} \{\frac{\partial \overline{v(i)v(m)}}{\partial \xi(m)} + [D(im) + D(mm) + \Gamma(nnm)]\ \overline{v(i)v(m)}$$

$$+ \Gamma(imn)\ \overline{v(m)v(n)}\} \quad \text{(no sum on i)} \tag{4-7}$$

Therefore, by utilizing equations (3-33), the Reynolds equations for turbulent flow become

$$\frac{\partial V(i)}{\partial t} + V(m)\frac{\partial V(i)}{\partial \xi(m)} + D(im)\ V(m)\ V(i) + \Gamma(imn)\ V(m)\ V(n) + \frac{\partial \overline{v(i)v(m)}}{\partial \xi(m)}$$

$$+ [D(im) + D(mm) + \Gamma(nnm)]\ \overline{v(i)\ v(m)} + \Gamma(imn)\ \overline{v(m)\ v(n)} =$$

$$- g(im)\frac{\partial}{\partial \xi(m)}(\frac{P}{\rho}) + \nu\{\nabla^2 V(i) + 2E(im)\frac{\partial V(i)}{\partial \xi(m)} + 2C(imn)\frac{\partial V(n)}{\partial \xi(m)}$$

$$+ P(i)\ V(i) + Q(in)\ V(n)\} - F(i) = 0 \quad \text{(no sum on i)} \tag{4-8}$$

Due to the presence of the six unknown Reynolds stresses $\overline{v(i)v(m)}$, the continuity and three momentum equations (4-8) are no longer sufficient to determine the mean flow. It is, therefore, necessary to furnish expressions for the unknown Reynolds stresses so that a closure of the equations can be achieved.

IV.3. Equations for the Reynolds Stresses

In addition to the Reynolds equations (4-8) for the mean motion, a parallel set of equations which describe the transport of Reynolds stresses can be derived using the following scheme:

$$\overline{v^i M^j + v^j M^i}$$

where M^i represents the ith instantaneous momentum equation. The resulting transport equations for the Reynolds stresses are

$$\frac{\partial}{\partial t}(\overline{v^i v^j}) + V^m(\overline{v^i v^j})_{,m} + \overline{v^j v^m}\, V^i_{,m} + \overline{v^i v^m}\, V^j_{,m} + (\overline{v^i v^j v^m})_{,m}$$
$$= - \overline{[g^{jm}(\frac{v^i p'}{\rho}) + g^{im}(\frac{v^j p'}{\rho})]}_{,m} + \overline{\frac{p'}{\rho}(g^{im} v^j_{,m} + g^{jm} v^i_{,m})}$$
$$+ \nu g^{mn}\, \overline{(v^i v^j_{,mn} + v^j v^i_{,mn})} \qquad (4\text{-}9)$$

The first two terms make up the material derivative (time and convective derivatives) following the mean flow. The following two terms represent the production of $\overline{v^i v^j}$ by the rate of strain of the fluid. The covariant derivative of the triple correlation terms stands for turbulent diffusion resulting from the convective motion of the turbulence. The first term on the right hand side of the equation is turbulent diffusion due to pressure fluctuations. It is often grouped together with the triple velocity correlation term. The next term is usually called the "pressure-strain" or "tendency-towards-isotropy" term. It leads to the redistribution of energy among the six Reynolds stresses $\overline{v^i v^j}$ and an approach towards the statistically most probable state, in which the turbulence is isotropic. The last term involving the viscosity represents the transport of energy by molecular action (viscous diffusion) as well as its dissipation into heat.

If we expand the covariant derivatives and introduce the physical components of velocity and geometric coefficients, the transport equations (4-9) can be expressed as

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left[\overline{v(i)v(j)}\right] + V(m)\,\frac{\partial}{\partial \xi(m)}\left[\overline{v(i)v(j)}\right] + [D(im) + D(jm)]\; V(m)\; \overline{v(i)v(j)} \\
&+ \Gamma(imn)\; V(m)\; \overline{v(j)v(n)} + \Gamma(jmn) V(m)\; \overline{v(i)v(n)} \\
&+ \overline{v(j)v(m)}\left[\frac{\partial V(i)}{\partial \xi(m)} + D(im)\; V(i) + \Gamma(imn)\; V(n)\right] \\
&+ \overline{v(i)v(m)}\left[\frac{\partial V(j)}{\partial \xi(m)} + D(jm)\; V(j) + \Gamma(jmn)\; V(n)\right] \\
&+ \frac{\partial}{\partial \xi(m)}\left[\overline{v(i)v(j)v(m)}\right] + [D(im) + D(jm) + D(mm) + \Gamma(nnm)]\; \overline{v(i)v(j)v(m)} \\
&+ \Gamma(imn)\; \overline{v(j)v(m)v(n)} + \Gamma(jmn)\; \overline{v(i)v(m)v(n)} \\
&= -\left\{g(jm)\,\frac{\partial}{\partial \xi(m)}\left[\frac{\overline{v(i)p'}}{\rho}\right] + g(im)\,\frac{\partial}{\partial \xi(m)}\left[\frac{\overline{v(j)p'}}{\rho}\right]\right\} \\
&- \left[E(ij)\,\frac{\overline{v(i)p'}}{\rho} + E(ji)\,\frac{\overline{v(j)p'}}{\rho}\right] - [C(ijn) + C(jin)]\,\frac{\overline{v(n)p'}}{\rho} \\
&+ \overline{\frac{p'}{\rho}\left[g(im)\,\frac{\partial v(j)}{\partial \xi(m)} + g(jm)\,\frac{\partial v(i)}{\partial \xi(m)}\right]} + \overline{\left[E(ij)\,\frac{v(i)p'}{\rho} + E(ji)\,\frac{v(j)p'}{\rho}\right]} \\
&+ [C(ij n) + C(jin)]\,\frac{\overline{v(n)p'}}{p} \\
&+ \nu\{\overline{v(i)\;\nabla^2 v(j)} + \overline{v(j)\;\nabla^2 v(i)} \\
&+ 2E(jm)\;\overline{v(i)\,\frac{\partial v(j)}{\partial \xi(m)}} + 2E(im)\;\overline{v(j)\,\frac{\partial v(i)}{\partial \xi(m)}} \\
&+ 2C(imn)\;\overline{v(j)\,\frac{\partial v(n)}{\partial \xi(m)}} + 2C(jmn)\;\overline{v(i)\,\frac{\partial v(n)}{\partial \xi(m)}} \\
&+ [P(i) + P(j)]\;\overline{v(i)v(j)} + Q(in)\;\overline{v(j)v(n)} + Q(jn)\;\overline{v(i)v(n)}\,\} \quad \text{(no sum on } i \text{ or } j\text{)}
\end{aligned}
\tag{4-10}
$$

The physical interpretation of the various terms in the above equation is much the same as it was for equation (4-9). For clarity, however, we shall examine some of the terms in greater detail. It is clear that the first term is the time derivative of $\overline{v(i)v(j)}$, and the next four terms represent the convective derivatives which describe the transport of $\overline{v(i)v(j)}$ by the mean flow. The following two terms again represent the production of $\overline{v(i)v(j)}$ by the mean

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rate of strain in the fluid. It is noted that some quantities in the production terms have exactly the same form as those which appeared in the convective terms although their origin is quite different. In the modeling of Reynolds stress equations, it is important to use the correct expression for the production term in order to retain the physical significance of the modeled terms. As mentioned before, terms involving triple velocity correlations represent turbulent diffusion due to the velocity fluctuations.

On the right hand side of equation (4-10), the first three terms make up turbulent diffusion due to the pressure fluctuations, while the following three terms involving pressure fluctuations are the pressure-strain terms. It is important to note that all pressure-velocity correlations without derivatives will cancel upon summation. However, the first term alone does not represent the total turbulent diffusion due to the pressure fluctuations. Similarly, the fourth term itself does not make up the overall pressure-strain term. The pressure-strain terms in general curvilinear coordinates do not possess a clear physical significance similar to that described in Nash and Patel (1972) for orthogonal coordinates since $\overline{v(i)v(i)}$ (unsummed) no longer represents the turbulent stresses normal to the surfaces of a fluid element. The redistribution of energy and approach towards isotropy should be interpreted by examining the turbulent kinetic energy, which is a contraction of the Reynolds stresses.

It is seen that the transport equations (4-10) contain many additional unknown quantities such as triple correlations and pressure-strain terms. Approximations must be made either at this or other levels for the unknown correlation terms in order to obtain a closure of the governing transport equations.

IV.4. Turbulent Kinetic Energy Equation

A quantity often used in turbulence closure schemes is the turbulent kinetic energy defined by

$$k = \frac{1}{2}\,\underset{\sim}{v}\cdot\underset{\sim}{v} = \frac{1}{2}\,\overline{v^i v_i} = \frac{1}{2}\,g_{ij}\,\overline{v^i v^j} \tag{4-11}$$

or

$$k = \frac{1}{2}\cos\theta_{ij}\,\overline{v(i)v(j)} \tag{4-12}$$

A transport equation for turbulent kinetic energy can be derived by contracting (multiplying by $\frac{1}{2} g_{ij}$) equation (4-9):

$$\frac{\partial k}{\partial t} + V^m k_{,m} + g_{ij} \overline{v^j v^m} V^i_{,m} + (\frac{1}{2} g_{ij} \overline{v^i v^j v^m})_{,m} + (\frac{\overline{v^m p'}}{\rho})_{,m} - \nu g^{mn} k_{,mn} + \nu g_{ij} g^{mn} \overline{v^i_{,m} v^j_{,n}} = 0 \qquad (4\text{-}13)$$

The first two terms represent the material derivative following the mean motion. The next term stands for the production of turbulent kinetic energy by the mean rate of strain, or the extraction of mean flow energy by the turbulence. The next three terms represent diffusion due to turbulent velocity fluctuations, pressure fluctuations and molecular action, respectively. The last term involving viscosity represents the decrease of turbulent kinetic energy due to viscous dissipation. Note that this is equal to the true dissipation (3-44) only for homogeneous turbulence.

Note that the contraction of the pressure-strain term is identical to zero, by virtue of the equation of continuity, even in a general curvilinear coordinate system. In other words, there is no net contribution from the pressure-strain terms to the turbulent kinetic energy. This clearly suggests that the main role of the pressure-strain term is in the redistribution of energy towards isotropy.

When physical components of V^i and v^i are used, the transport equation for k becomes

$$\frac{\partial k}{\partial t} + V(m) \frac{\partial k}{\partial \xi(m)} + \cos \theta_{ij} \overline{v(j)v(m)} V(i,m) + \overline{v(m) \frac{\partial}{\partial \xi(m)} [\frac{1}{2} \cos \theta_{ij} v(i)v(j)]}$$

$$+ \frac{\partial}{\partial \xi(m)} [\frac{\overline{v(m)p'}}{\rho}] + [D(mm) + \Gamma(nnm)] \frac{\overline{v(m)p'}}{\rho} - \nu \nabla^2 k + \varepsilon = 0 \qquad (4\text{-}14)$$

where the last term is the viscous dissipation

$$\varepsilon = \nu \cos \theta_{ij} g(mn) \overline{v(i,m)v(j,n)} \qquad (4\text{-}15)$$

and V(i,m) and v(i,m) are the physical components of the covariant derivative $V^i_{,m}$ and $v^i_{,m}$, i.e.,

$$V(i,m) = g_{ii}^{1/2}\ g_{mm}^{-1/2}\ V^i_{,m} = \frac{\partial V(i)}{\partial \xi(m)} + D(im)\ V(i) + \Gamma(ims)\ V(s) \tag{4-16}$$

Although it is possible to group the turbulent diffusion terms due to the pressure fluctuations in the following form using the equation of continuity (4-4):

$$\frac{\partial}{\partial \xi(m)} \left[\frac{\overline{v(m)p'}}{\rho}\right] - \overline{\frac{p'}{\rho} \frac{\partial v(m)}{\partial \xi(m)}} \tag{4-17}$$

the second term, however, does not represent the pressure-strain term. In the turbulence modeling of the k equation, it is therefore incorrect to model these two terms separately. Similarly, the triple velocity correlation term in equation (4-14) should not be split into

$$\frac{\partial}{\partial \xi(m)} \left[\frac{1}{2} \cos\theta_{ij}\ \overline{v(i)\ v(j)\ v(m)}\right] - \frac{1}{2} \cos\theta_{ij}\ \overline{v(i)\ v(j)\ \frac{\partial v(m)}{\partial \xi(m)}} \tag{4-18}$$

and modeled independently since both terms represent only part of the turbulent diffusion due to the velocity fluctuations.

IV.5. Equation for Viscous Dissipation ε

A transport equation for the dissipation term ε defined in (4-15) can be derived using the following scheme:

$$\nu g_{ij}\ g^{rs}\ \overline{[v^j_{,s}\ M^i_{,r} + v^i_{,r}\ M^j_{,s}]} \tag{4-19}$$

where, as before, M^i represents the ith instantaneous momentum equation.

The resulting transport equation is

$$\frac{\partial \varepsilon}{\partial t} + V^m \varepsilon_{,m} = -\left(\nu g_{ij} g^{rs}\ \overline{v^m v^i_{,r} v^j_{,s}} + \frac{2\nu}{\rho}\ g^{rs}\ \overline{v^m_{,s}\ p'_{,r}} - \nu g^{mn}\ \varepsilon_{,n}\right)_{,m}$$

$$-2\nu\ \left(g_{ij}\ g^{rs}\ \overline{v^i_{,m} v^j_{,s}} + g_{mj} g^{si}\ \overline{v^j_{,s} v^r_{,i}}\right)\ V^m_{,r} - 2\nu g_{ij} g^{rs}\ \overline{v^m v^j_{,s}}\ V^i_{,mr}$$

$$- 2\nu\, g_{ij} g^{rs}\, \overline{v^j_{,s} v^m_{,r} v^i_{,m}} - 2\nu^2\, g_{ij} g^{rs} g^{mn}\, \overline{v^i_{,mr} v^j_{,ns}} \qquad (4\text{-}20)$$

It is noted again that the transport equation for ε involves many additional unknown quantities which can not be expressed in terms of lower-order correlation by simply deriving their corresponding transport equations. In the well known k-ε turbulence model, all terms on the right hand side of equation (4-20), except the molecular diffusion term $\nu g^{mn} \varepsilon_{,mn} = \nu \nabla^2 \varepsilon$, are modeled to obtain a closure equation.

The transport equation for ε can be written in terms of the physical components by utilizing the expressions given in equation (2-53) for covariant derivatives and the usual definitions for physical components of tensor quantities. No such attempt has been made here, not only due to the obvious complexity of the result but also because most closure schemes use only modeled forms of this equation.

V. EDDY-VISCOSITY AND k-ε MODEL EQUATIONS

The classical Boussinesq eddy-viscosity model hypothesizes that the Reynolds stresses are linearly related to the mean rate of strain. The relationship is similar to that derived for the viscous stresses:

$$- \overline{v^i v^j} = 2\nu_t e^{ij} - \frac{2}{3} g^{ij} k \qquad (5\text{-}1)$$

where ν_t denotes the isotropic eddy viscosity, which depends upon the local flow conditions. The turbulent kinetic energy term has been added to make the equation valid when contracted (multiplied by g_{ij}):

$$-g_{ij}\overline{v^i v^j} = 2\nu_t e^i_i - 2k = - 2k$$

since $e^i_i = 0$ for an incompressible fluid. By replacing the contravariant strain rate with

$$\begin{aligned} e^{ij} &= g^{\ell i} g^{kj} e_{k\ell} \\ &= g^{\ell i} g^{kj} \frac{1}{2} (V_{k,\ell} + V_{\ell,k}) \end{aligned}$$

further simplifications yield

$$- \overline{v^i v^j} = \nu_t \, (g^{\ell i} V^j_{,\ell} + g^{kj} V^i_{,k}) - \frac{2}{3} g^{ij} k \qquad (5\text{-}2)$$

If an algebraically-specified eddy-viscosity model is used, substitution of the above equation into the momentum equation (4-6) completes the closure problem.

In the k-ε model, the eddy viscosity is related to the turbulent kinetic energy and its rate of dissipation by

$$\nu_t = c_\mu \frac{k^2}{\varepsilon} \qquad (5\text{-}3)$$

Since k and ε are scalars which are invariant with respect to coordinate transformation, each term in the curvilinear-tensor form of their transport equations has a direct counterpart in the Cartesian-tensor form of these equations. This allows one to simply transform each term of the modeled Cartesian tensor equations to the curvilinear tensor form. It is assumed that the model constants are not affected by the coordinate transformation.

The k-ε model equations in Cartesian tensor form (see, for example, Bradshaw, Cebeci, and Whitelaw (1981)) are:

$$\frac{\partial k}{\partial t} + V_i \frac{\partial k}{\partial x^i} = \frac{\partial}{\partial x^j} \left[\left(\nu + \frac{\nu_t}{\sigma_k}\right) \frac{\partial k}{\partial x^j}\right] + G - \varepsilon \tag{5-4}$$

$$\frac{\partial \varepsilon}{\partial t} + V_i \frac{\partial \varepsilon}{\partial x^i} = \frac{\partial}{\partial x^j} \left[\left(\nu + \frac{\nu_t}{\sigma_\varepsilon}\right) \frac{\partial \varepsilon}{\partial x^j}\right] + C_{\varepsilon 1} \frac{\varepsilon}{k} G - C_{\varepsilon 2} \frac{\varepsilon^2}{k} \tag{5-5}$$

where the generation term is

$$G = - \overline{v_i v_j} \frac{\partial V_i}{\partial x^j} = \nu_t \left(\frac{\partial V_i}{\partial x^j} + \frac{\partial V_j}{\partial x^i}\right) \frac{\partial V_i}{\partial x^j} \tag{5-6}$$

These may be directly converted to the curvilinear tensor form:

$$\frac{\partial k}{\partial t} + V^m k_{,m} = \left[\left(\nu + \frac{\nu_t}{\sigma_k}\right) g^{mn} k_{,n}\right]_{,m} + G - \varepsilon \tag{5-7}$$

$$\frac{\partial \varepsilon}{\partial t} + V^m \varepsilon_{,m} = \left[\left(\nu + \frac{\nu_t}{\sigma_\varepsilon}\right) g^{mn} \varepsilon_{,n}\right]_{,m} + C_{\varepsilon 1} \frac{\varepsilon}{k} G - C_{\varepsilon 2} \frac{\varepsilon^2}{k} \tag{5-8}$$

with

$$G = - g_{ij} \overline{v^j v^m} V^i_{,m} = \nu_t \left(V^i_{,m} V^m_{,i} + g_{ij} g^{mn} V^i_{,m} V^j_{,n}\right) \tag{5-9}$$

Note that the convection and diffusion terms in these equations have the same general form as the energy equation (3-45), and the generation term G is similar to the dissipation function (3-44).

Introducing the eddy-viscosity relation (5-2) into the Reynolds equations (4-9), and including the continuity equation, gives the following system of six coupled, nonlinear partial differential equations:

$$V^i_{,i} = 0 \tag{5-10}$$

$$\frac{\partial V^i}{\partial t} + (V^m - g^{mn}\nu_{t,n})V^i_{,m} - \nu_{t,n}g^{im}V^n_{,m}$$

$$= -g^{im}(\frac{p}{\rho} + \frac{2}{3}k)_{,m} + (\nu + \nu_t)\, g^{mn}V^i_{,mn} \tag{5-11}$$

$$\frac{\partial k}{\partial t} + (V^m - \frac{1}{\sigma_k}g^{mn}\nu_{t,n})k_{,m} = (\nu + \frac{\nu_t}{\sigma_k})\, g^{mn}k_{,mn} + G - \varepsilon \tag{5-12}$$

$$\frac{\partial \varepsilon}{\partial t} + (V^m - \frac{1}{\sigma_\varepsilon}g^{mn}\nu_{t,n})\varepsilon_{,m} = (\nu + \frac{\nu_t}{\sigma_\varepsilon})\, g^{mn}\,\varepsilon_{,mn} + C_{\varepsilon 1}\frac{\varepsilon}{k}G - C_{\varepsilon 2}\frac{\varepsilon^2}{k} \tag{5-13}$$

where the generally accepted model constants are: $C_\mu = 0.09$, $C_{\varepsilon 1} = 1.44$, $C_{\varepsilon 2} = 1.92$, $\sigma_k = 1.0$, and $\sigma_\varepsilon = 1.3$. These equations are valid for incompressible turbulent flow in any coordinate system. No approximations have been made except for those inherent in the turbulence model. Together with the proper boundary conditions, equations (5-10) through (5-13) constitute a well-closed problem which, in theory, may be solved for the six unknowns: V^i, p, k, and ε.

Equations (5-10) through (5-13) can also be expanded in terms of their physical components to yield:

$$\frac{1}{g^{1/2}}\frac{\partial}{\partial \xi^i}\{g^{1/2}\, g_{ii}^{-1/2}\, V(i)\} = 0 \tag{5-14}$$

$$\frac{\partial V(i)}{\partial t} + V(m)\frac{\partial V(i)}{\partial \xi(m)} + D(im)\, V(m)\, V(i) + \Gamma\, (imn)\, V(m)\, V(n)$$

$$- \frac{\partial \nu_t}{\partial \xi(n)}\{g(mn)\frac{\partial V(i)}{\partial \xi(m)} + g(im)\frac{\partial V(n)}{\partial \xi(m)}$$

$$+ E(in)\, V(i) + E(ni)\, V(n) - [C(ins) + C(nis)]\, V(s)\}$$

$$= - g(im)\frac{\partial}{\partial \xi(m)}(\frac{p}{\rho} + \frac{2}{3}k) + (\nu + \nu_t)\{\nabla^2 V(i)$$

$$+ 2E(im)\frac{\partial V(i)}{\partial \xi(m)} + 2C(imn)\frac{\partial V(n)}{\partial \xi(m)}$$

$$+ P(i)\, V(i) + Q(in)\, V(n)\} \tag{5-15}$$

$$\frac{\partial k}{\partial t} + [V(m) - \frac{1}{\sigma_k} g(mn) \frac{\partial \nu_t}{\partial \xi(n)}] \frac{\partial k}{\partial \xi(m)} = (\nu + \frac{\nu_t}{\sigma_k}) \nabla^2 k + G - \varepsilon \qquad (5\text{-}16)$$

$$\frac{\partial \varepsilon}{\partial t} + [V(m) - \frac{1}{\sigma_\varepsilon} g(mn) \frac{\partial \nu_t}{\partial \xi(n)}] \frac{\partial \varepsilon}{\partial \xi(m)} = (\nu + \frac{\nu_t}{\sigma_\varepsilon}) \nabla^2 \varepsilon + C_{\varepsilon 1} \frac{\varepsilon}{k} G - C_{\varepsilon 2} \frac{\varepsilon^2}{k} \qquad (5\text{-}17)$$

where the turbulence generation term is

$$G = \nu_t [V(i,m)\ V(m,i) + \cos \theta_{ij}\ g(mn)\ V(i,m)\ V(j,n)] \qquad (5\text{-}18)$$

VI. REDUCTION OF EQUATIONS TO ORTHOGONAL COORDINATES

One way to illustrate the generality of the equations in curvilinear tensor form is to simplify them to the more familiar equations in general orthogonal coordinates given by Nash and Patel (1972). The physical components of velocity must be used when attempting to compare equations derived by tensor methods with those developed using orthogonal curvilinear coordinates. This is clearly evident because the equations given by Nash and Patel are written using the physical velocity components. This exercise also provides an example of the use of tensor analysis and the introduction of physically measurable vector components into the tensor equations.

VI.1. Simplified Tensor Operations

It seems useful at this point to reiterate several of the tensor relations which take simpler forms when the coordinate system is orthogonal. Of prime importance is the metric tensor and its inverse:

$$\begin{aligned} g_{ii} &= h_i^2 \\ g^{ii} &= \frac{1}{h_i^2} \\ g_{ij} &= g^{ij} = 0 \text{ for } i \neq j \end{aligned} \tag{6-1}$$

The determinant of the covariant metric tensor or Jacobian is

$$g = J^2 = (h_i h_j h_k)^2$$

Recall that (2-56) gave the Christoffel symbols for an orthogonal coordinate system as

$$\Gamma^i_{ii} = \frac{1}{h_i} \frac{\partial h_i}{\partial \xi^i}$$

$$\Gamma^i_{ij} = \Gamma^i_{ji} = \frac{1}{h_i} \frac{\partial h_i}{\partial \xi^j}$$

$$\Gamma^i_{jj} = -\frac{h_j}{h_i^2}\frac{\partial h_j}{\partial \xi^i}$$

$$\Gamma^i_{jk} = 0$$

where i,j,k are in cyclic order. The physical components of a tensor, for instance the contravariant velocity vector, can be found from

$$V(i) = \sqrt{g_{ii}}\, V^i = h_i V^i \text{ (no sum on i)} \tag{6-2}$$

Nash and Patel found it useful to introduce the curvature parameter

$$K_{ij} = \frac{1}{h_i h_j}\frac{\partial h_i}{\partial \xi^j} \tag{6-3}$$

which, for $i \neq j$ represents the curvature of the surface j=constant in the direction i. K_{ij} can be related to the physical components of the Christoffel symbols in the following manner:

$$\Gamma(iii) = K_{ii} \tag{6-4a}$$

$$\Gamma(iij) = \Gamma(iji) = K_{ij} \tag{6-4b}$$

$$\Gamma(ijj) = -K_{ji} \tag{6-4c}$$

$$\Gamma(ijk) = 0 \tag{6-4d}$$

where i,j,k are in cyclic order. The other geometric coefficients can also be simplified by noting that

$$g(ii) = 1 \tag{6-5a}$$

$$g(ij) = 0 \tag{6-5b}$$

Therefore, D(im), E(im), P(i), Q(in) etc. can be expressed in terms of the Christoffel symbols or curvature parameters as

$$D(im) = -\Gamma(imi) = -K_{im} \tag{6-6}$$

$$E(im) = D(im) = -K_{im} \tag{6-7}$$

$$C(imn) = \Gamma(imn) \tag{6-8}$$

$$P(i) + Q(ii) = C(imj)\Gamma(jmi) + C(imk)\Gamma(kmi)$$
$$= -K_{ij}{}^2 - K_{ji}{}^2 - K_{ik}{}^2 - K_{ki}{}^2 = \alpha_{ii} \tag{6-9}$$

$$Q(ij) = \frac{1}{h_i}\frac{\partial k_{ij}}{\partial \xi^i} - \frac{1}{h_j}\frac{\partial k_{ji}}{\partial \xi^j} - K_{kj}\,(K_{ki} + K_{ji}) + K_{ki}K_{ij} = \alpha_{ij} \tag{6-10}$$

$$Q(ik) = \frac{1}{h_i}\frac{\partial K_{ik}}{\partial \xi^i} - \frac{1}{h_k}\frac{\partial K_{ki}}{\partial \xi^k} - K_{jk}\,(K_{ki} + K_{ji}) + K_{ji}K_{ik} = \alpha_{ik} \tag{6-11}$$

$$f(m) = -\Gamma(mnn) \tag{6-12}$$

and the Laplacian operator for scalar quantities (A-14) can be written as

$$\nabla^2 = \frac{1}{h_m^2}\frac{\partial^2}{\partial\xi^m\partial\xi^m} + f(m)\frac{\partial}{\partial\xi(m)}$$

$$= \frac{1}{h_i^2}\frac{\partial^2}{\partial\xi^i\partial\xi^i} + \frac{1}{h_j^2}\frac{\partial^2}{\partial\xi^j\partial\xi^j} + \frac{1}{h_k^2}\frac{\partial^2}{\partial\xi^k\partial\xi^k} + (-K_{ii} + K_{ji} + K_{ki})\frac{1}{h_i}\frac{\partial}{\partial\xi^i}$$

$$+ (K_{ij} - K_{jj} + K_{kj})\frac{1}{h_j}\frac{\partial}{\partial\xi^j} + (K_{ik} + K_{jk} - K_{kk})\frac{1}{h_k}\frac{\partial}{\partial\xi^k} \tag{6-13}$$

In the above equations i,j,k are in cyclic order.

VI.2. Continuity Equation

By substituting the geometric coefficients into equations (4-3) and (4-4), the equation of continuity can be written in both the conservative and nonconservative forms in orthogonal coordinates as

$$\frac{\partial}{\partial\xi^i}[h_j h_k\ V(i)] + \frac{\partial}{\partial\xi^j}[h_i h_k\ V(j)] + \frac{\partial}{\partial\xi^k}[h_i h_j\ V(k)] = 0 \tag{6-14}$$

or

$$\frac{1}{h_i}\frac{\partial V(i)}{\partial \xi^i} + \frac{1}{h_j}\frac{\partial V(j)}{\partial \xi^j} + \frac{1}{h_k}\frac{\partial V(k)}{\partial \xi^k} + (K_{ji} + K_{ki})\, V(i) + (K_{ij} + K_{kj})\, V(j)$$

$$+ (K_{ik} + K_{jk})\, V(k) = 0 \qquad (6\text{-}15)$$

where i,j,k are in cyclic order. The reader may verify that this result is identical to the continuity equation (2.11) given by Nash and Patel with (i,j,k) = (1,2,3) and V(i) = U,V,W.

VI.3. Momentum (Reynolds) Equations

If we introduce the geometric coefficients of (6-1) through (6-13) into the Reynolds equations (4-8) and assume F(i) = 0, the mean momentum equations for turbulent flow can be written as

$$\frac{\partial V(i)}{\partial t} + \frac{V(i)}{h_i}\frac{\partial V(i)}{\partial \xi^i} + \frac{V(j)}{h_j}\frac{\partial V(i)}{\partial \xi^j} + \frac{V(k)}{h_k}\frac{\partial V(i)}{\partial \xi^k} + [K_{ij}V(i) - K_{ji}V(j)]\, V(j)$$

$$+ [K_{ik}V(i) - K_{ki}V(k)]\, V(k) + \frac{1}{h_i}\frac{\partial}{\partial \xi^i}[\overline{v(i)v(i)} + \frac{p}{\rho}] + \frac{1}{h_j}\frac{\partial}{\partial \xi^j}[\overline{v(i)v(j)}]$$

$$+ \frac{1}{h_k}\frac{\partial}{\partial \xi^k}[\overline{v(i)v(k)}] + (K_{ji} + K_{ki})\,\overline{v(i)v(i)} - K_{ji}\,\overline{v(j)v(j)} - K_{ki}\,\overline{v(k)v(k)}$$

$$+ (2K_{ij} + K_{kj})\,\overline{v(i)v(j)} + (2K_{ik} + K_{jk})\,\overline{v(i)v(k)} - \nu\{\nabla^2 V(i)$$

$$+ 2K_{ij}\frac{1}{h_i}\frac{\partial V(j)}{\partial \xi^i} - 2K_{ji}\frac{1}{h_j}\frac{\partial V(j)}{\partial \xi^j} + 2K_{ik}\frac{1}{h_i}\frac{\partial V(k)}{\partial \xi^i} - 2K_{ki}\frac{1}{h_k}\frac{\partial V(k)}{\partial \xi^k}$$

$$+ \alpha_{ii}V(i) + \alpha_{ij}V(j) + \alpha_{ik}V(k)\} = 0 \qquad (6\text{-}16)$$

with i,j,k in cyclic order. Letting $(\xi^1, \xi^2, \xi^3) = (x,y,z)$ and (V(1), V(2), V(3)) = (U,V,W), and then rotating the indices i,j,k, equations (6-16) become the Reynolds equations (2.20) through (2.22) in Nash and Patel.

VI.4. Reynolds Stress Transport Equations

The transport equation (4-10) for the Reynolds stresses $\overline{v(i)v(j)}$ may be reduced to a form valid for an orthogonal coordinate system in the same manner as was done for the Reynolds equations. The result is:

$$
\begin{aligned}
&\frac{\partial}{\partial t}\,[\overline{v(i)v(j)}] + \frac{V(i)}{h_i}\frac{\partial}{\partial \xi^i}\,[\overline{v(i)v(j)}] + \frac{V(j)}{h_j}\frac{\partial}{\partial \xi^j}\,[\overline{v(i)v(j)}] + \\
&\frac{V(k)}{h_k}\frac{\partial}{\partial \xi^k}\,[\overline{v(i)v(j)}] + [K_{ji}V(i) - K_{ij}V(j)]\,\overline{v(i)v(i)} + [K_{ij}V(i) - K_{ji}V(j)]\overline{v(j)v(j)} \\
&+ [K_{jk}V(j) - K_{kj}V(k)]\,\overline{v(i)v(k)} + [K_{ik}V(i) - K_{ki}V(k)]\,\overline{v(j)v(k)} \\
&+ [\frac{1}{h_i}\frac{\partial V(j)}{\partial \xi^i} - K_{ij}\,V(i)]\,\overline{v(i)v(i)} + [\frac{1}{h_j}\frac{\partial V(i)}{\partial \xi^j} - K_{ji}V(j)]\,\overline{v(j)v(j)} \\
&+ [\frac{1}{h_i}\frac{\partial V(i)}{\partial \xi^i} + K_{ij}\,V(j) + K_{ik}\,V(k) + \frac{1}{h_j}\frac{\partial V(j)}{\partial \xi^j} + K_{ji}V(i) + K_{jk}V(k)]\,\overline{v(i)v(j)} \\
&+ [\frac{1}{h_k}\frac{\partial V(j)}{\partial \xi^k} - K_{kj}\,V(k)]\,\overline{v(i)v(k)} + [\frac{1}{h_k}\frac{\partial V(i)}{\partial \xi^k} - K_{ki}\,V(k)]\,\overline{v(j)v(k)} \\
&+ \frac{1}{h_i}\frac{\partial}{\partial \xi^i}\,[\overline{v(i)v(i)v(j)}] + \frac{1}{h_j}\frac{\partial}{\partial \xi^j}\,[\overline{v(i)v(j)v(j)}] + \frac{1}{h_k}\frac{\partial}{\partial \xi^k}\,[\overline{v(i)v(j)v(k)}] \\
&+ (2K_{ji} + K_{ki})\,\overline{v(i)v(i)v(j)} + (2K_{ij} + K_{kj})\,\overline{v(i)v(j)v(j)} \\
&- K_{ij}\,\overline{v(i)v(i)v(i)} - K_{ji}\,\overline{v(j)v(j)v(j)} - K_{ki}\,\overline{v(j)v(k)v(k)} - K_{kj}\,\overline{v(i)v(k)v(k)} \\
&+ 2(K_{ik} + K_{jk})\,\overline{v(i)v(j)v(k)} + \frac{1}{h_i}\frac{\partial}{\partial \xi^i}\,[\frac{\overline{v(j)p'}}{\rho}] + \frac{1}{h_j}\frac{\partial}{\partial \xi^j}\,[\frac{\overline{v(i)p'}}{\rho}] \\
&- \overline{\frac{p'}{\rho}\,[\frac{1}{h_i}\frac{\partial v(j)}{\partial \xi^i} + \frac{1}{h_j}\frac{\partial v(i)}{\partial \xi^j}]} - \nu\{\overline{v(i)\nabla^2 v(j)} + \overline{v(j)\nabla^2 v(i)} + 2K_{ij}\,\overline{\frac{v(j)}{h_i}\frac{\partial v(j)}{\partial \xi^i}} \\
&- 2K_{ji}\,\overline{\frac{v(j)}{h_j}\frac{\partial v(j)}{\partial \xi^j}} + 2K_{ik}\,\overline{\frac{v(j)}{h_i}\frac{\partial v(k)}{\partial \xi^i}} - 2K_{ki}\,\overline{\frac{v(j)}{h_k}\frac{\partial v(k)}{\partial \xi^k}} - 2K_{ij}\,\overline{\frac{v(i)}{h_i}\frac{\partial v(i)}{\partial \xi^i}} \\
&+ 2K_{ji}\,\overline{\frac{v(i)}{h_j}\frac{\partial v(i)}{\partial \xi^j}} + 2K_{jk}\,\overline{\frac{v(i)}{h_j}\frac{\partial v(k)}{\partial \xi^j}} - 2K_{kj}\,\overline{\frac{v(i)}{h_k}\frac{\partial v(k)}{\partial \xi^k}} + (\alpha_{ii} + \alpha_{jj})\,\overline{v(i)v(j)} \\
&+ \alpha_{ik}\,\overline{v(j)v(k)} + \alpha_{jk}\overline{v(i)v(k)} + \alpha_{ji}\overline{v(i)v(i)} + \alpha_{ij}\overline{v(j)v(j)}\} = 0 \qquad (6\text{-}17)
\end{aligned}
$$

These equations are identical to equations (2.28) through (2.30) for $\overline{uv}$ ($i=1$, $j=2$, $k=3$), $\overline{uw}$ ($i=3, j=1, k=2$) and $\overline{vw}$ ($i=2, j=3, k=1$) given by Nash and Patel if the equation of continuity is employed to rearrange the triple velocity correlations.

The transport equation for $\frac{1}{2}\overline{v(i)v(i)}$ in a general orthogonal system can be derived in a similar manner by letting i=j in equations (4-10):

$$\frac{\partial}{\partial t}\left[\frac{1}{2}\overline{v(i)v(i)}\right] + \frac{V(i)}{h_i}\frac{\partial}{\partial \xi^i}\left[\frac{1}{2}\overline{v(i)v(i)}\right] + \frac{V(j)}{h_j}\frac{\partial}{\partial \xi^j}\left[\frac{1}{2}\overline{v(i)v(i)}\right]$$

$$+ \frac{V(k)}{h_k}\frac{\partial}{\partial \xi^k}\left[\frac{1}{2}\overline{v(i)v(i)}\right] + [K_{ij}V(i)-K_{ji}V(j)]\,\overline{v(i)v(j)}+[K_{ik}V(i)-K_{ki}V(k)]\,\overline{v(i)v(k)}$$

$$+ \overline{v(i)v(i)}\left[\frac{1}{h_i}\frac{\partial V(i)}{\partial \xi^i} + K_{ij}V(j) + K_{ik}V(k)\right] + \overline{v(i)v(j)}\left[\frac{1}{h_j}\frac{\partial V(i)}{\partial \xi^j} - K_{ji}V(j)\right]$$

$$+\overline{v(i)v(k)}\left[\frac{1}{h_k}\frac{\partial V(i)}{\partial \xi^k} -K_{ki}V(k)\right]+ \frac{1}{h_i}\frac{\partial}{\partial \xi^i}\left[\frac{1}{2}\overline{v(i)v(i)v(i)}\right]+ \frac{1}{h_j}\frac{\partial}{\partial \xi^j}\left[\frac{1}{2}\overline{v(i)v(i)v(j)}\right]$$

$$+ \frac{1}{h_k}\frac{\partial}{\partial \xi^k}\left[\frac{1}{2}\overline{v(i)v(i)v(k)}\right] + \frac{1}{2}(K_{ji}+ K_{ki})\,\overline{v(i)v(i)v(i)} + (\frac{3}{2}K_{ij}$$

$$+ K_{kj})\,\overline{v(i)v(i)v(j)} + (\frac{3}{2}K_{ik} + K_{jk})\,\overline{v(i)v(i)v(k)} - K_{ji}\,\overline{v(i)v(j)v(j)}$$

$$- K_{ki}\overline{v(i)v(k)v(k)} + \frac{1}{h_i}\frac{\partial}{\partial \xi^i}\left[\frac{\overline{v(i)p'}}{\rho}\right] - \overline{\frac{p'}{\rho}\frac{1}{h_i}\frac{\partial v(i)}{\partial \xi^i}} - \nu\{\overline{v(i)\nabla^2 v(i)}$$

$$+ 2K_{ij}\,\overline{\frac{v(i)}{h_i}\frac{\partial v(j)}{\partial \xi^i}} - 2K_{ji}\,\overline{\frac{v(i)}{h_j}\frac{\partial v(j)}{\partial \xi^j}} + 2K_{ik}\,\overline{\frac{v(i)}{h_i}\frac{\partial v(k)}{\partial \xi^i}} - 2K_{ki}\,\overline{\frac{v(i)}{h_k}\frac{\partial v(k)}{\partial \xi^k}}$$

$$+ \alpha_{ii}\,\overline{v(i)v(i)} + \alpha_{ij}\,\overline{v(i)v(j)} + \alpha_{ik}\,\overline{v(i)v(k)}\} = 0 \qquad (6\text{-}18)$$

When the continuity equation is employed, equation (6-18) reduces to equations (2.24) through (2-26) in Nash and Patel for $\overline{u^2}$, $\overline{v^2}$, and $\overline{w^2}$.

VII. EVALUATION OF TYPICAL CHRISTOFFEL SYMBOLS FOR NUMERICALLY-GENERATED COORDINATES

For most practical applications requiring the use of curvilinear tensor equations, it is most unlikely that an analytical coordinate system will be available. The required geometric coefficients must usually be evaluated from numerically-generated coordinates. Naturally the question is asked as to what sort of geometric coefficients are produced from a typical grid generation method. To illustrate the answer a few typical Christoffel symbols for two, ship-like three dimensional bodies will be shown. The magnitude and variation of the Christoffel symbols are of interest because of their ubiquity in the equations of fluid motion.

The two coordinate systems considered are those for the 3:1 elliptical cross-section body of Groves et al. (1982) and the SSPA cargo ship hull of Larsson (1974). Flow computations for these bodies using partially-tansformed, partially-parabolic equations in numerically-generated, body-fitted coordinates, have been carried out by Chen and Patel (1985a,b). The Christoffel symbols shown herein were calculated from the numerical coordinates used by Chen and Patel in their computations and may be considered as representative of the geometric coefficients which may be encountered in practice.

The numerical coordinate grids for these bodies were generated by using the Poisson equation technique described by Chen and Patel (1985a). The coordinate system and general orientation of the base vectors are shown in figure 3. Different views of the coordinate grids for the 3:1 body and the SSPA hull are shown in figures 4 and 5, respectively.

The relationship between the Christoffel symbols and the base vectors can be readily identified by comparing (2-52) and (2-58), from which one concludes that

$$\frac{\partial \underset{\sim}{a}_j}{\partial \xi^k} = \Gamma^i_{jk} \underset{\sim}{a}_i \qquad (7\text{-}1)$$

Hence, the Christoffel symbol represents the rate of change of the jth base vector with respect to the kth coordinate resolved along the ith coordinate direction. The Christoffel symbols can be calculated from (2-54):

$$\Gamma^i_{jk} = \frac{1}{2} g^{i\ell}\left(\frac{\partial g_{\ell j}}{\partial \xi^k} + \frac{\partial g_{k\ell}}{\partial \xi^j} - \frac{\partial g_{jk}}{\partial \xi^\ell}\right)$$

Evidently, the Christoffel symbols depend upon the metric coefficients and their derivatives. The g_{ij}, g^{ij}, and Γ^i_{jk} were calculated using standard finite-difference techniques. Derivatives were evaluated using central differences wherever possible and with forward or backward differences on the domain boundaries.

Figures 6a and 6b show the variation of Γ^2_{13} on the $\eta = 2,3$ coordinate surfaces of the 3:1 body. Forward differencing seems to be the cause of the jagged variation on $\eta = 2$. The variation of Γ^2_{13} is much smoother on the next coordinate surface, $\eta = 3$, where central differencing can be used to calculate the necessary derivatives. Recall that, for an orthogonal coordinate system, Γ^2_{13} is zero. The influence of the rapid variation of the body geometry is shown in figures 6c through 6h. Near the body Γ^3_{13} and Γ^2_{11} show more variation compared to the regions well removed from the body surface. This is particularly evident at $\xi = 40$, where the body terminates and the coordinate lines make a transition from elliptical to nearly circular shapes (see figures 4d and 4e). In the wake region the Christoffel symbols again have a smooth variation. Thus, whenever the geometry of the bounding surface changes rapidly one must expect equally rapid changes in the geometric coeffcients.

Further effects of the body geometry on the Christoffel symbols can be seen in figure 7 which shows the variation of Γ^3_{13} for the SSPA hull. Note that the hull terminates at $\xi = 20$. The variation of Γ^3_{13} in this case is qualitatively similar to that for the 3:1 body except near the hull surface.

Because flow calculations using the complete curvilinear tensor formulation of the governing equations have only just begun, it is not possible at this time to fully assess the influence of the magnitude and variation of the geometrical coefficients upon the calculations. If the so-called wall functions are used to satisfy the boundary conditions on the body for turbulent flows, the coordinate surface $\eta = 2$ is not used in the calculations. Clearly this alleviates some difficulties because the coefficients on $\eta = 3$ can be evaluated using central differencing and figure 6b shows that Γ^2_{13} has a smooth distribution. Trouble may arise if the equations must be integrated to the

wall, as will be the case for laminar flows, or if low-Reynolds-number turbulence models are employed. At the outer boundaries the problems may not be as severe. One can simply specify the boundary conditions 'one step' inside the domain boundary, unless of course computations are being done for a duct flow. Observe that the Christoffel symbols are not the only geometric coefficients which show up in the equations. Combinations of the metric tensor, Christoffel symbols, and derivatives of the Christoffel symbols appear throughout the equations. Obviously additional study is required to completely explore the influence of the geometric coefficients upon the quality of flow computations.

APPENDIX

Vector Operations in Tensor Form

The usual vector operations can be readily expressed in terms of curvilinear tensors using the methods outlined in Section II. The results are summarized below.

The scalar product of two vectors is

$$\underset{\sim}{A} \cdot \underset{\sim}{B} = g_{ij} A^i B^j \tag{A-1}$$

or in terms of physical components,

$$g_{ij} A^i B^j = \frac{g_{ij} A(i) B(j)}{\sqrt{g_{ii} g_{jj}}} = \cos \theta_{ij} \; A(i) \; B(j) \tag{A-2}$$

The vector product which yields a contravariant vector is

$$\underset{\sim}{A} \times \underset{\sim}{B} = \frac{\varepsilon^{ijk}}{g^{1/2}} A_j B_k = C^i \tag{A-3}$$

or, defining a covariant vector,

$$\underset{\sim}{A} \times \underset{\sim}{B} = g^{1/2} \; \varepsilon_{ijk} A^j B^k = C_i \tag{A-4}$$

The reader may verify that this reduces to

$$C(i) = \varepsilon_{ijk} \; A(j) B(k) \tag{A-5}$$

for an orthogonal coordinate system.

The gradient of a scalar function forms the components of the covariant vector given by

$$(\nabla \phi)_i = \frac{\partial \phi}{\partial \xi^i} \tag{A-6}$$

By introducing the physical components this becomes

$$\nabla\phi(i) = \sqrt{g_{ii}}\, g^{ij} \frac{\partial\phi}{\partial\xi^j} \quad \text{(no sum on i)} \tag{A-7}$$

for general coordinates, and reduces to

$$\nabla\phi(i) = \frac{1}{h_i} \frac{\partial\phi}{\partial\xi^i} \quad \text{(no sum on i)} \tag{A-8}$$

for orthogonal coordinates.

The divergence of a vector is obtained by contracting the covariant derivative of a contravariant vector,

$$\text{div } (\underset{\sim}{A}) = \nabla \cdot \underset{\sim}{A} = A^i_{\ ,i} = \frac{\partial A^i}{\partial\xi^i} + \Gamma^i_{ik} A^k \tag{A-9}$$

or, using (2-55),

$$\nabla\cdot\underset{\sim}{A} = \frac{1}{g^{1/2}} \frac{\partial}{\partial\xi^i} (g^{1/2} A^i) \tag{A-10}$$

The curl of a vector can be expressed by either

$$\nabla \times \underset{\sim}{A} = \frac{\varepsilon^{ijk}}{g^{1/2}} A_{j,k} \tag{A-11a}$$

or

$$\nabla \times \underset{\sim}{A} = \frac{\varepsilon^{ijk}}{g^{1/2}} g_{jp} A^p_{\ ,k} \tag{A-11b}$$

The Laplacian of a scalar function ϕ, defined as

$$\nabla^2\phi = \nabla\cdot(\nabla\phi) \tag{A-12}$$

is a scalar given by

$$\nabla^2\phi = g^{mn}\phi_{,mn} \tag{A-13}$$

or

$$\nabla^2\phi = (g_{mm}g_{nn})^{-1/2}\ g(mn)\ \frac{\partial^2\phi}{\partial\xi^m\partial\xi^n} + f(m)\ \frac{\partial\phi}{\partial\xi(m)} \tag{A-14}$$

with

$$g(mn) = (g_{mm}g_{nn})^{1/2} g^{mn} \tag{A-14a}$$

$$f(m) = -\ g(rs)\Gamma(mrs) \tag{A-14b}$$

$$\frac{\partial}{\partial\xi(m)} = g_{mm}^{-1/2}\ \frac{\partial}{\partial\xi^m} \tag{A-14c}$$

Substituting the physical components for the tensor components yields the following Laplacian of a vector:

$$\nabla^2\underset{\sim}{A} = g^{mn}\ A^i_{,mn} = g_{ii}^{-1/2}\ \{\nabla^2 A(i) + 2E(im)\ \frac{\partial A(i)}{\partial\xi(m)} + 2C(ims)\ \frac{\partial A(s)}{\partial\xi(m)}$$

$$+ P(i)A(i) + Q(is)A(s)\} \tag{A-15}$$

where

$$E(im) = g(mn)D(in) \tag{A-15a}$$

$$C(ims) = g(mn)\Gamma(ins) \tag{A-15b}$$

$$P(i) = g_{ii}^{1/2}\ \nabla^2 g_{ii}^{-1/2} = \frac{\partial E(im)}{\partial\xi(m)} + [D(im) + D(mm) + \Gamma(nnm)]\ E(im) \tag{A-15c}$$

$$Q(is) = \frac{\partial C(ims)}{\partial\xi(m)} + [D(im) + D(sm) + D(mm) + \Gamma(nnm)]\ C(ims)$$

$$+ C(imn)\ \Gamma(nms) \tag{A-15d}$$

$$D(im) = g_{ii}^{1/2}\ \frac{\partial g_{ii}^{-1/2}}{\partial\xi(m)} = \frac{-1}{2g_{ii}}\ \frac{\partial g_{ii}}{\partial\xi(m)} \tag{A-15e}$$

Green's theorem written in curvilinear tensor form is

$$\int_{V} A^{i}_{,i}\, dV = \int_{S} A^{i} n_{i} dS \tag{A-16}$$

where n_i is the outward unit normal to the surface S bounding the volume V. Other forms of Green's theorem are

$$\int_{V} g^{ij} \phi_{,ij} dV = \int_{S} \phi_{,i}\, n^{i} dS = \int_{S} \frac{\delta\phi}{\delta n}\, dS \tag{A-17}$$

where $\frac{\delta\phi}{\delta n}$ is the intrinsic derivative normal to the surface S.

Stokes' theorem is given by

$$\int_{S} g^{-1/2} \varepsilon^{ijk} A_{k,j} dS = \oint_{c} A_{k} \frac{dx^{k}}{ds}\, ds \tag{A-18}$$

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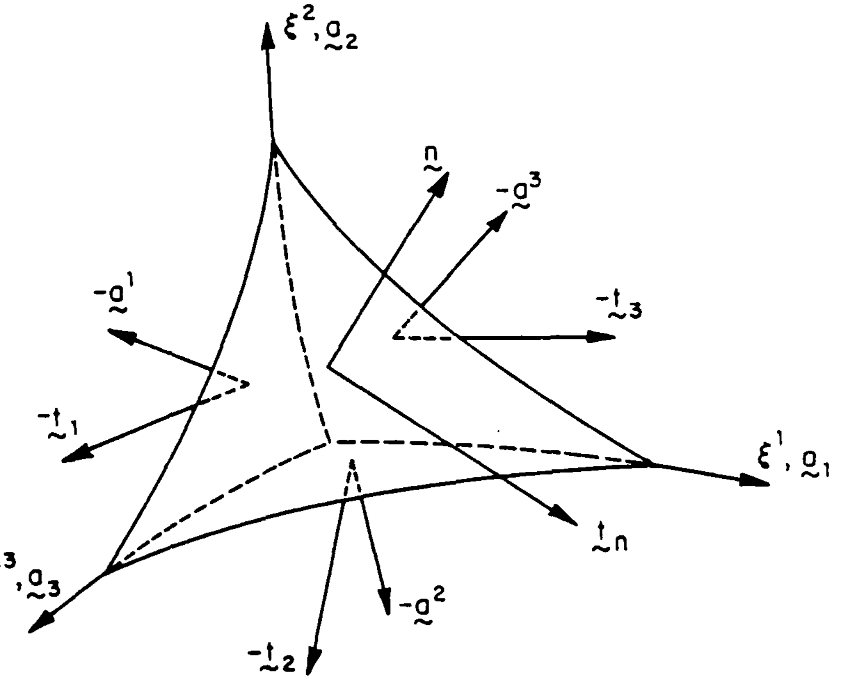

Figure 1. Orientation of the Covariant ($\underset{\sim}{a}_i$) and Contravariant ($\underset{\sim}{a}^j$) Base Vectors and Surface Stress Vectors.

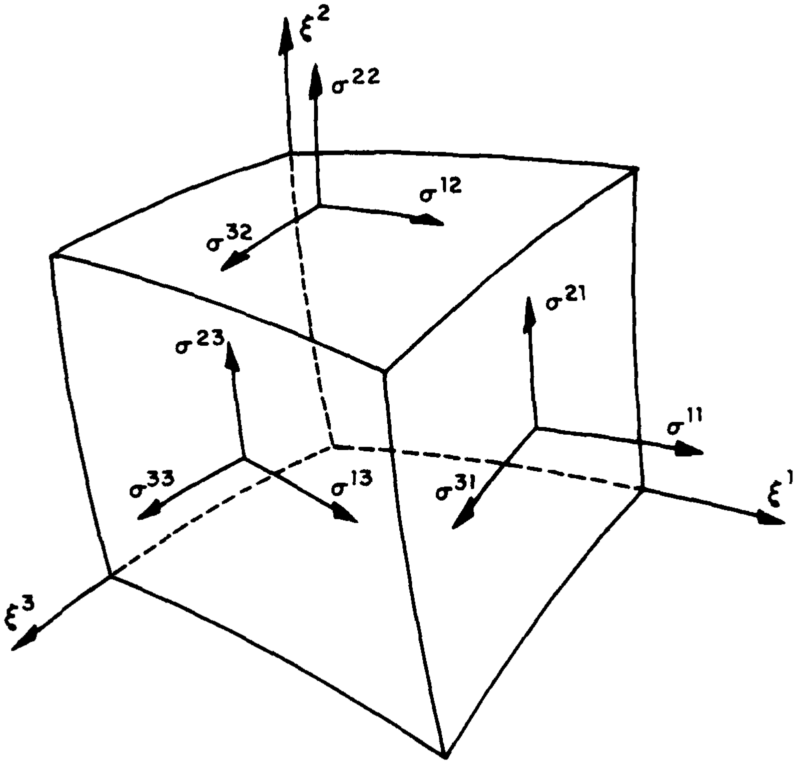

Figure 2. Components of the Stress Tensor.

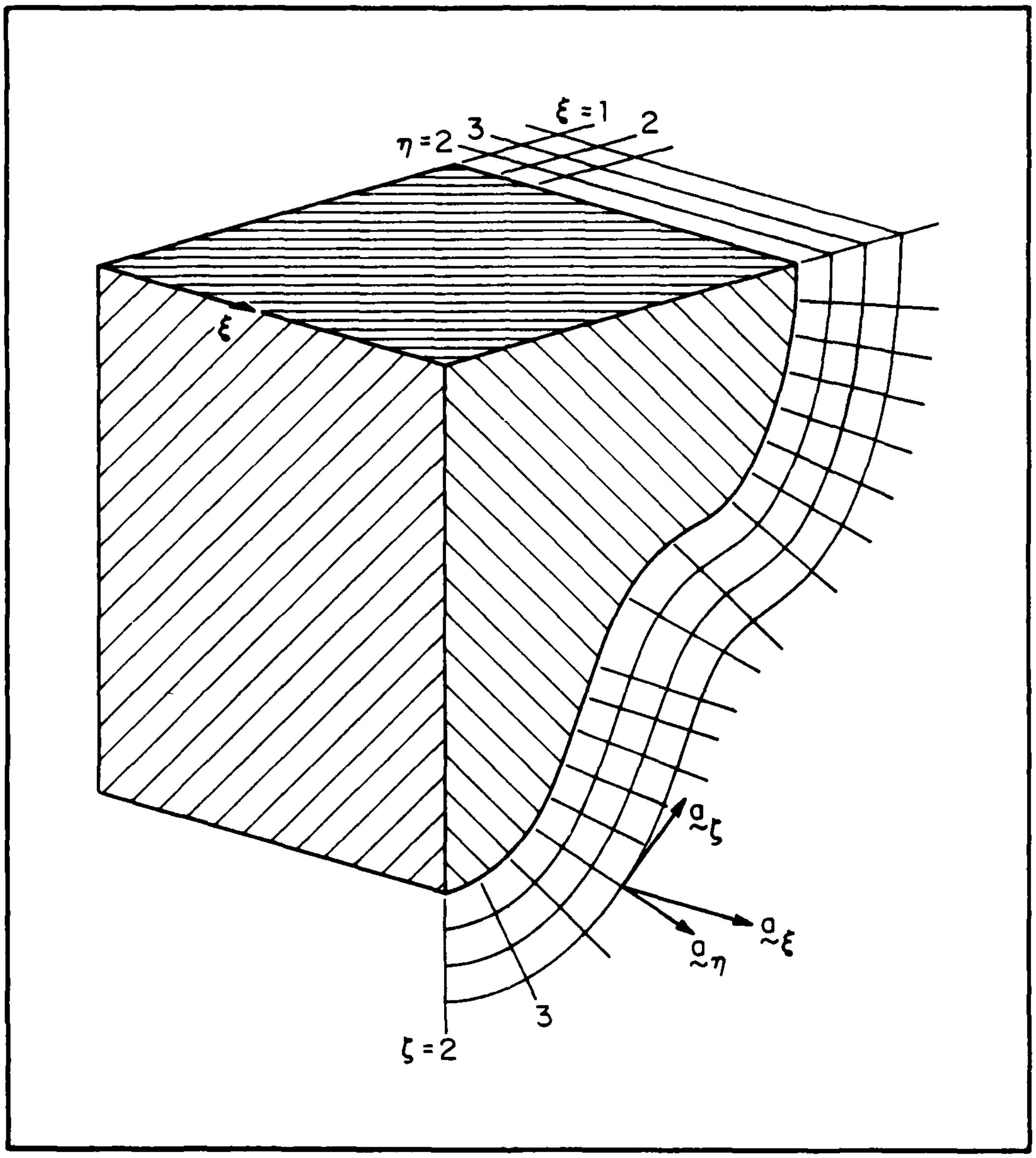

Figure 3. Coordinate System for Three-Dimensional Ship-Like Bodies.

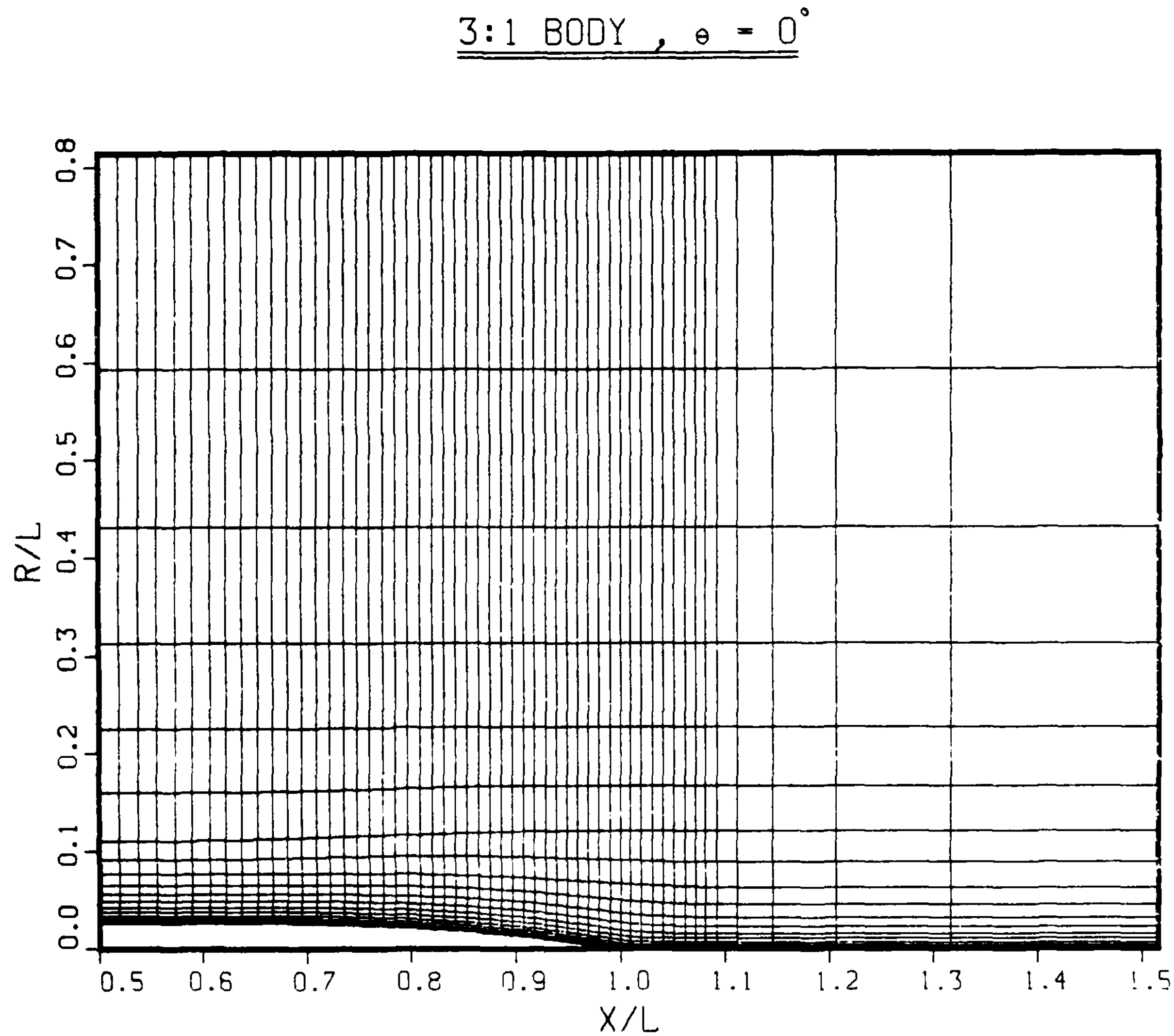

Figure 4. Numerical Grid for the 3:1 Body of Groves et al. (1982)
(a) Longitudinal View at $\theta = 0^{\circ}$.

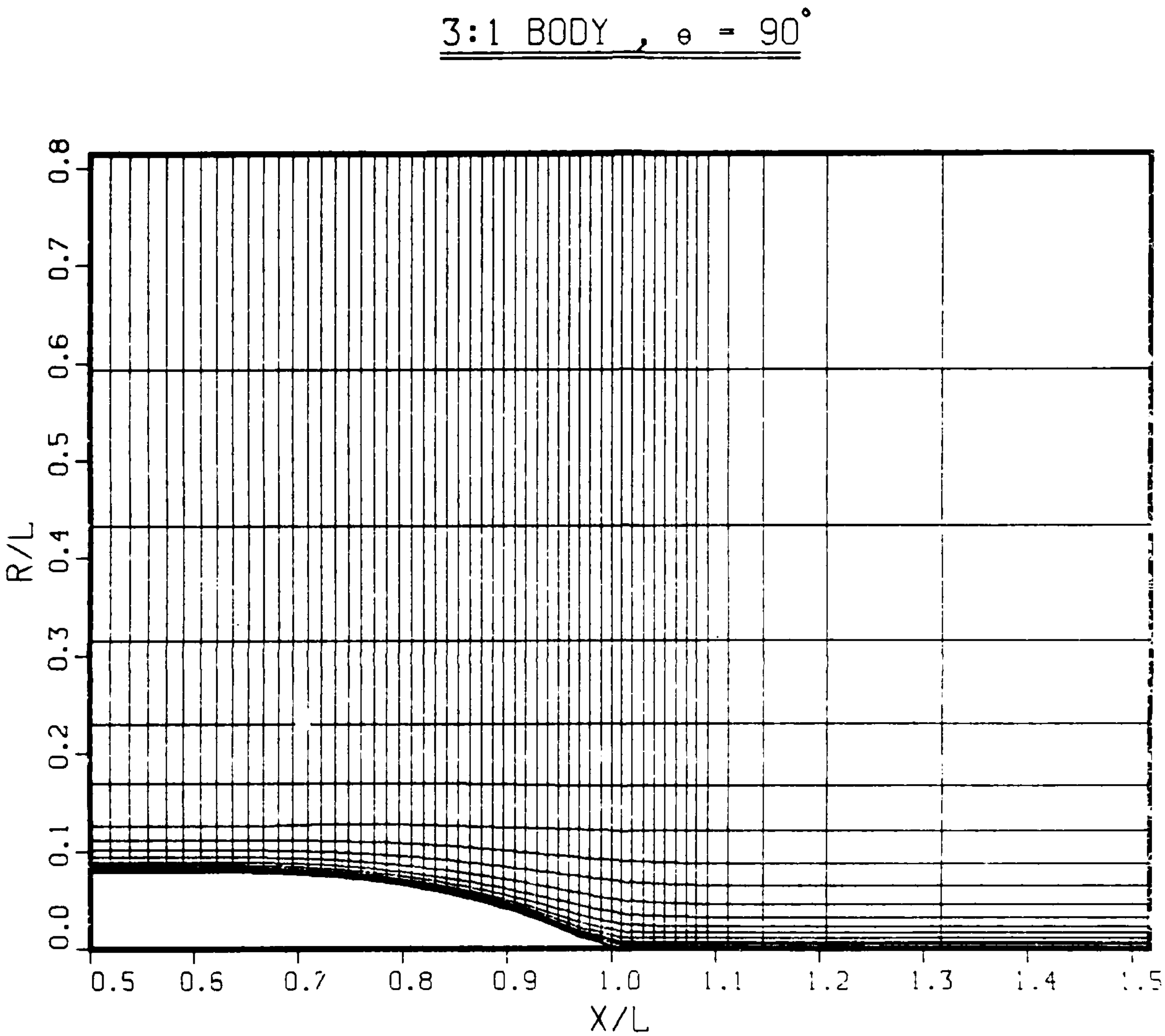

Figure 4. Continued
(b) Longitudinal View at $\theta = 90^{\circ}$.

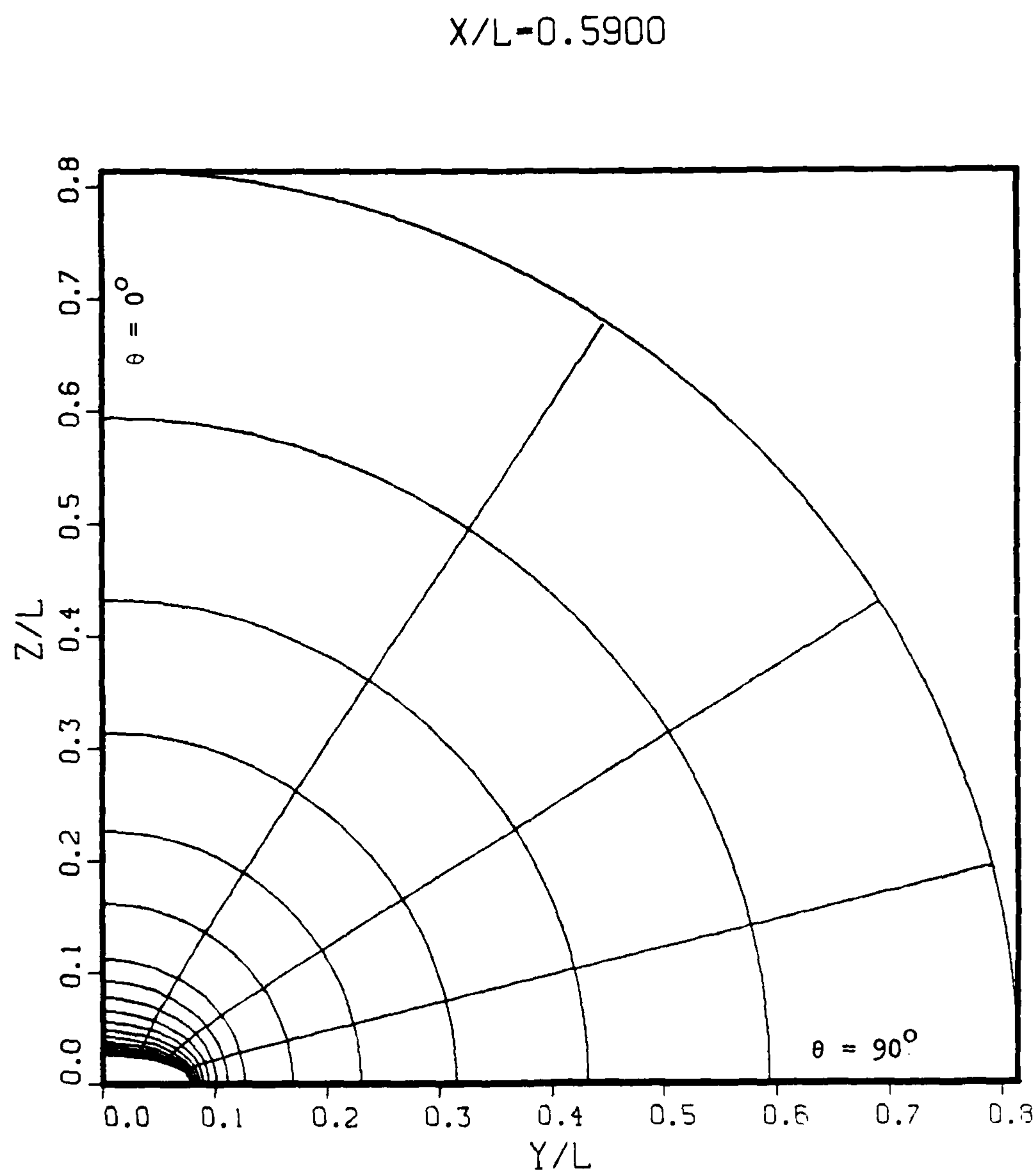

Figure 4. Continued
(c) Transverse View as x/L = 0.5900.

X/L-0.9071

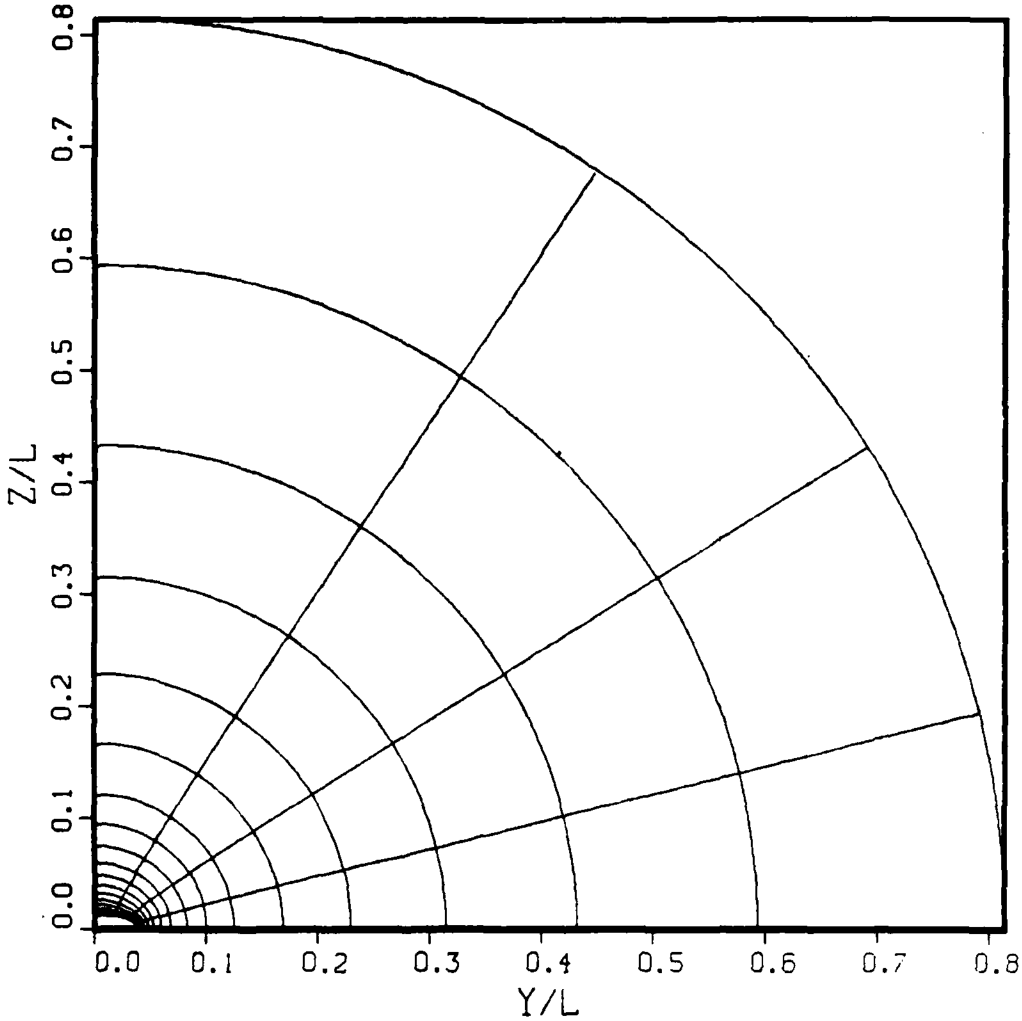

Figure 4. Continued
(d) Transverse View at x/L = 0.9071.

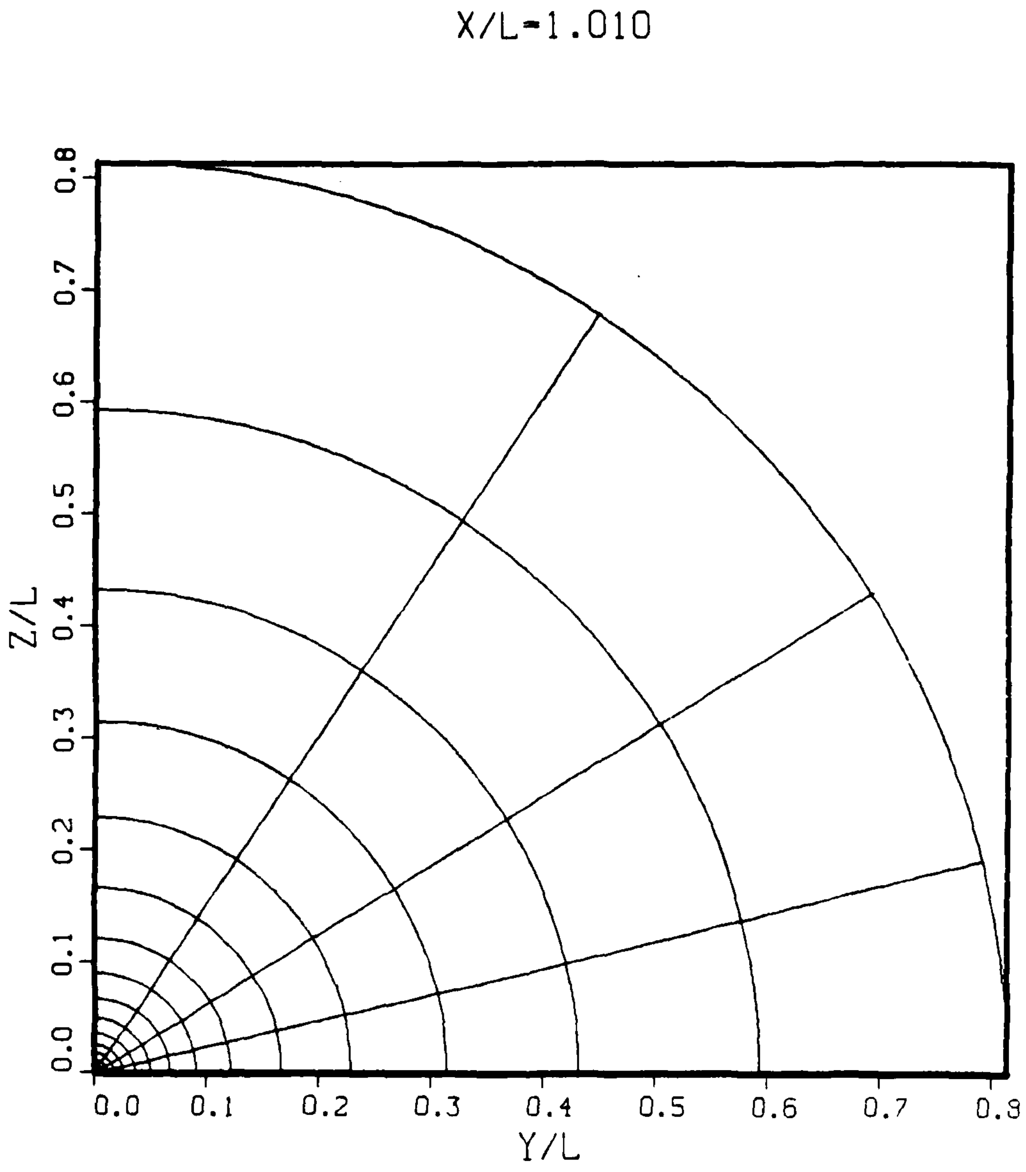

Figure 4. Continued
(e) Transverse View at x/L = 1.010.

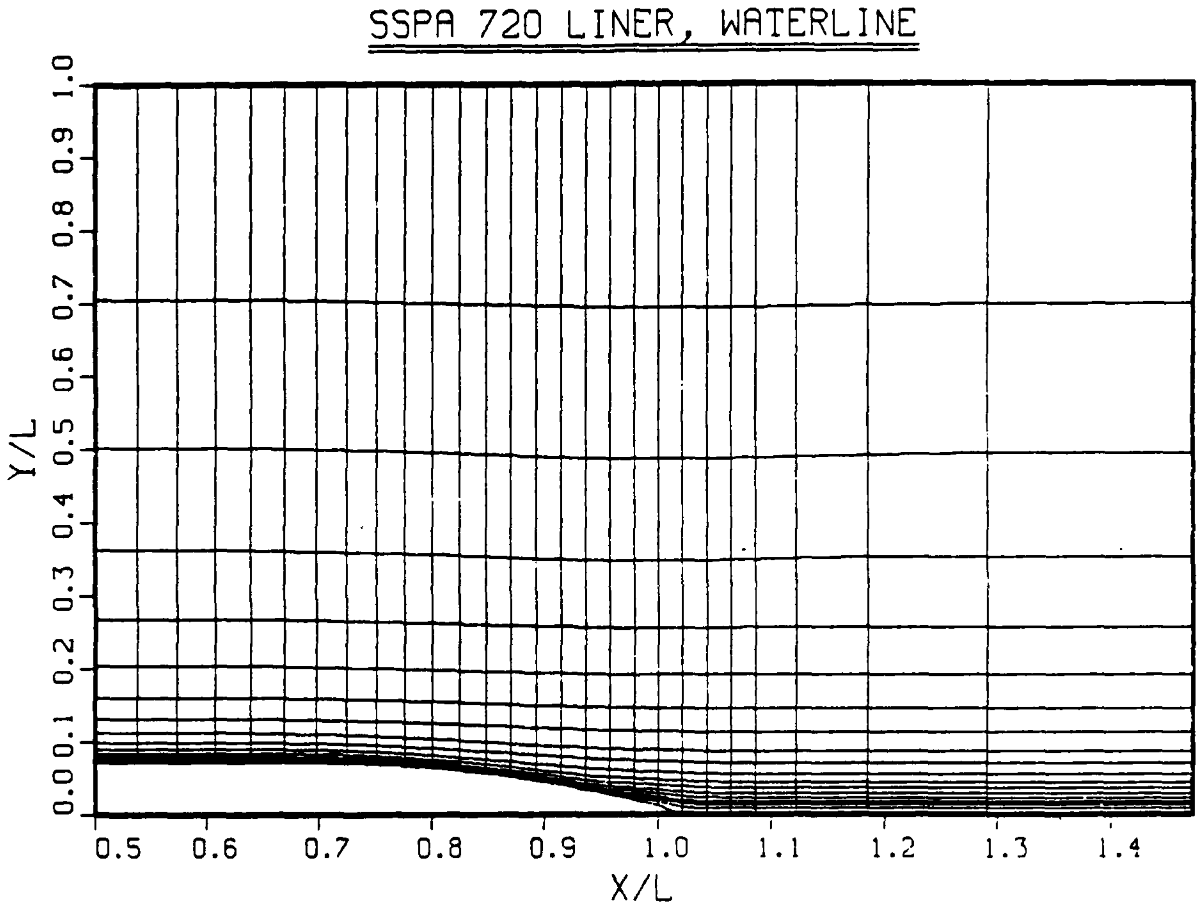

Figure 5. Numerical Grid for the SSPA 720 Liner of Larsson (1974)
(a) Waterline View.

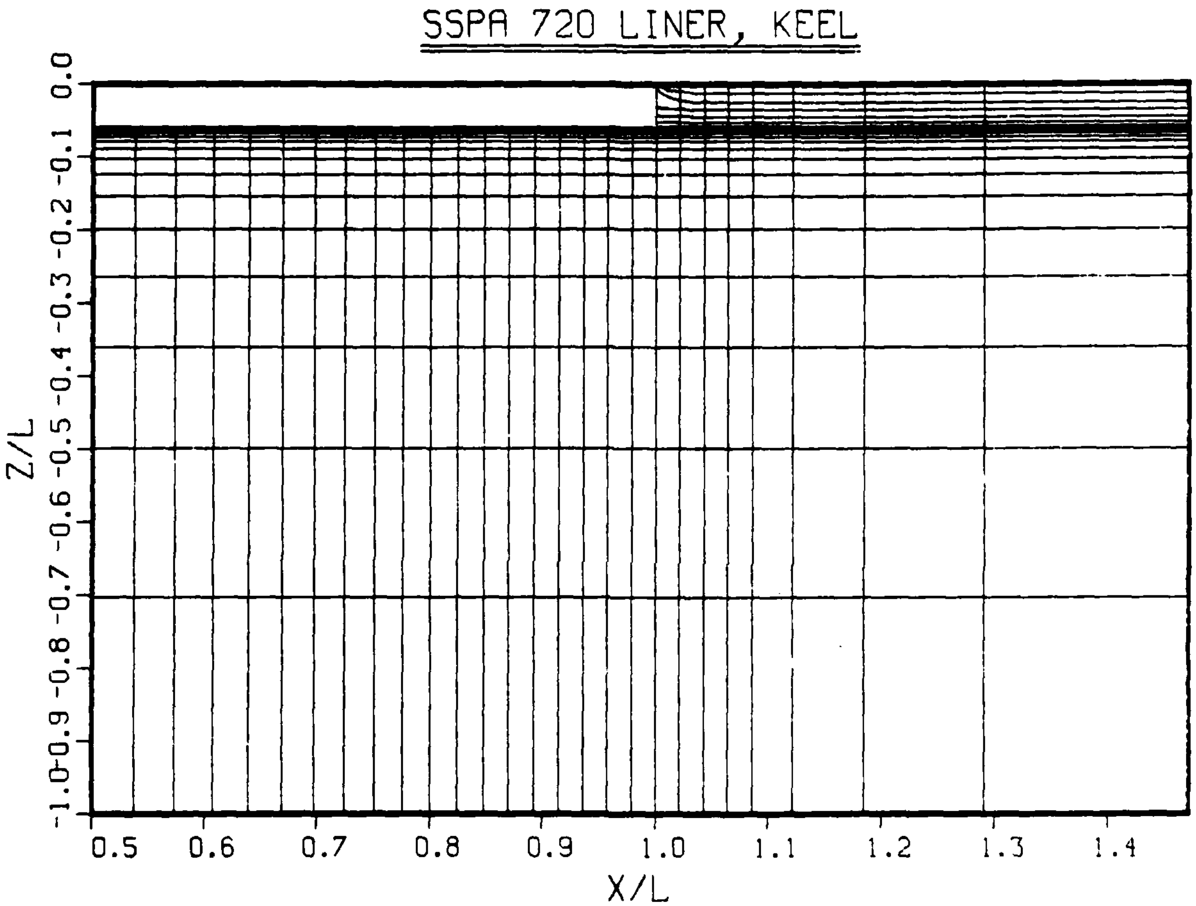

Figure 5. Continued
(b) Keel View.

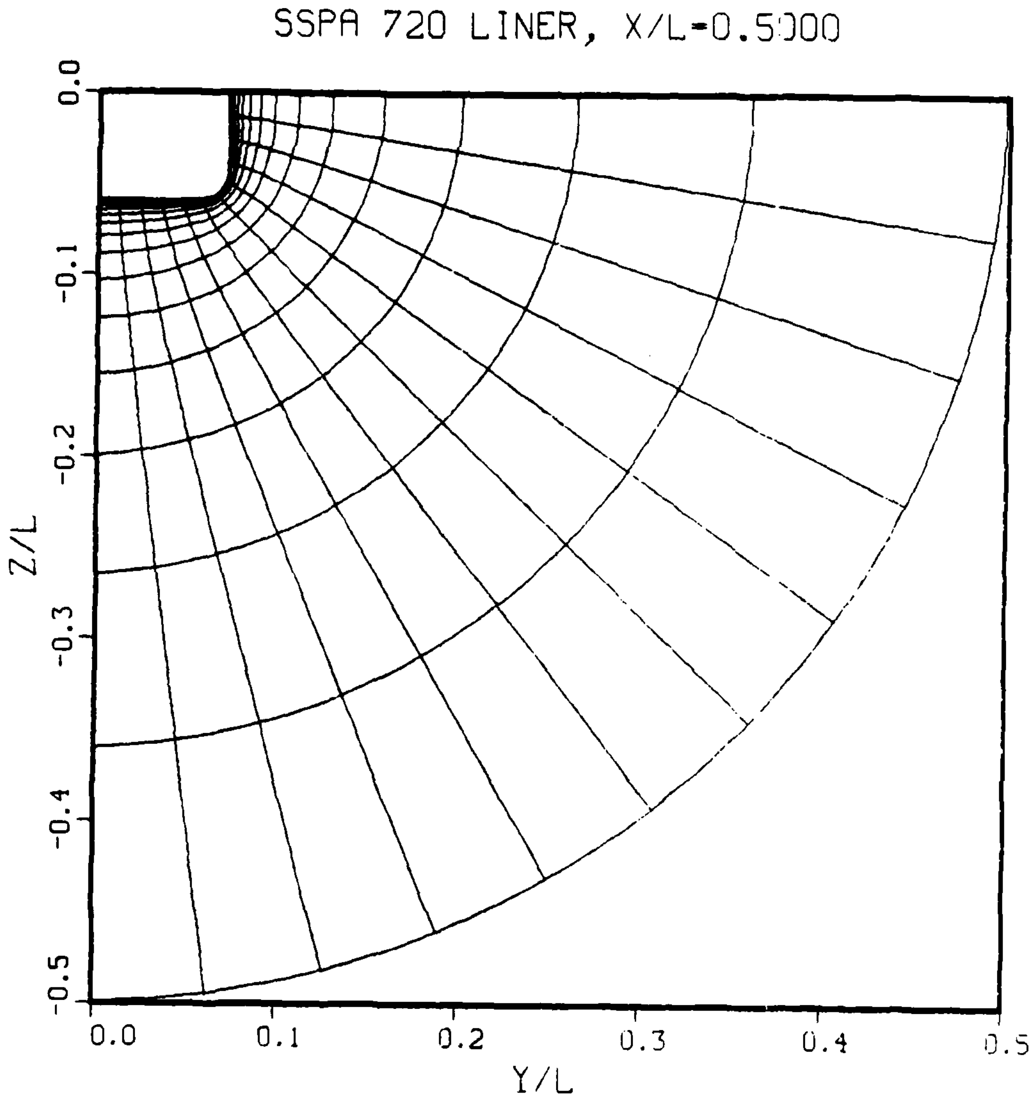

Figure 5. Continued
(c) Transverse View at x/L = 0.500.

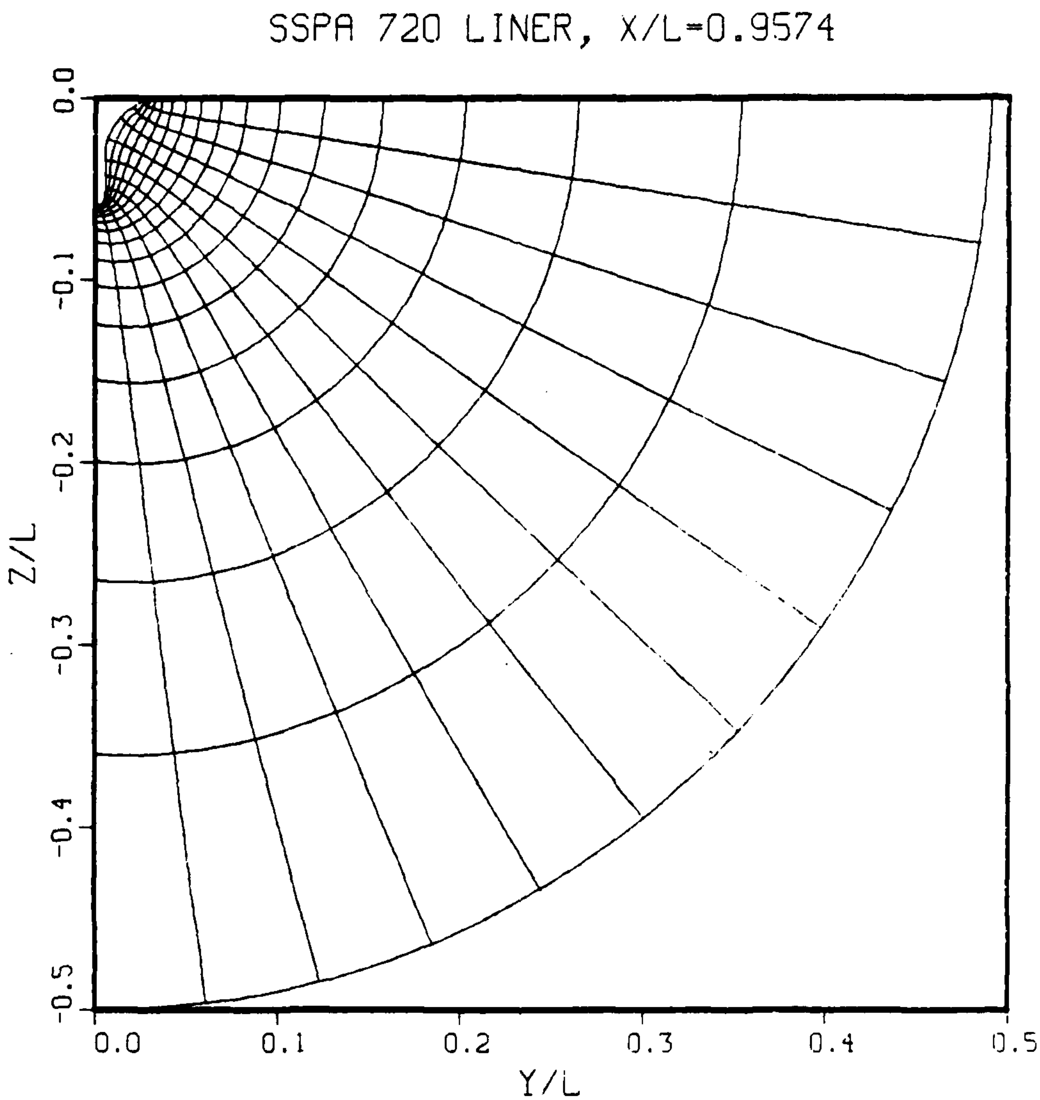

Figure 5. Continued
(d) Transverse View at x/L = 0.9574.

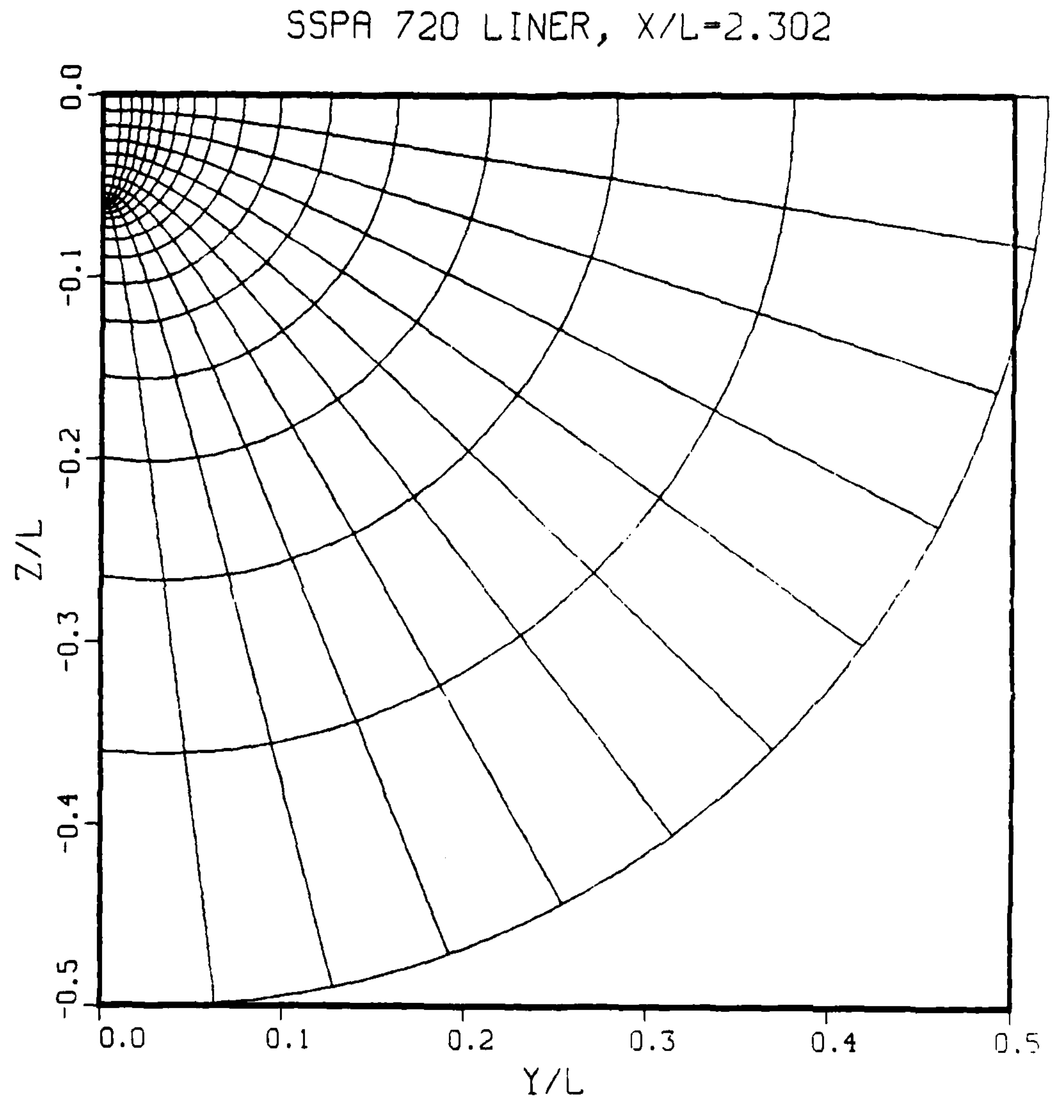

Figure 5. Continued
(e) Transverse View at x/L = 2.302.

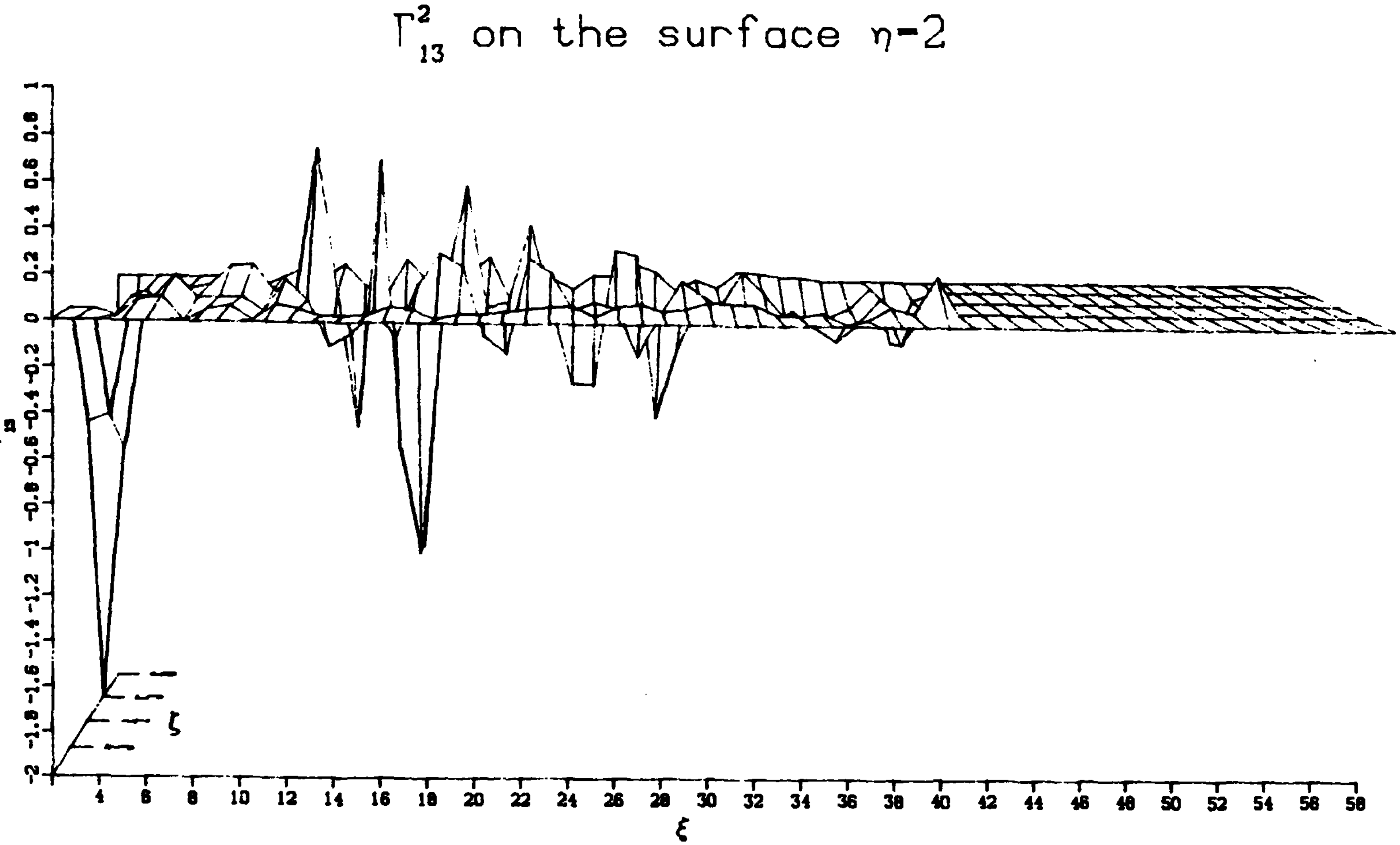

Figure 6. Calculated Christoffel Symbols for the 3:1 Elliptic Body

(a) Γ^2_{13} on the Surface $\eta = 2$.

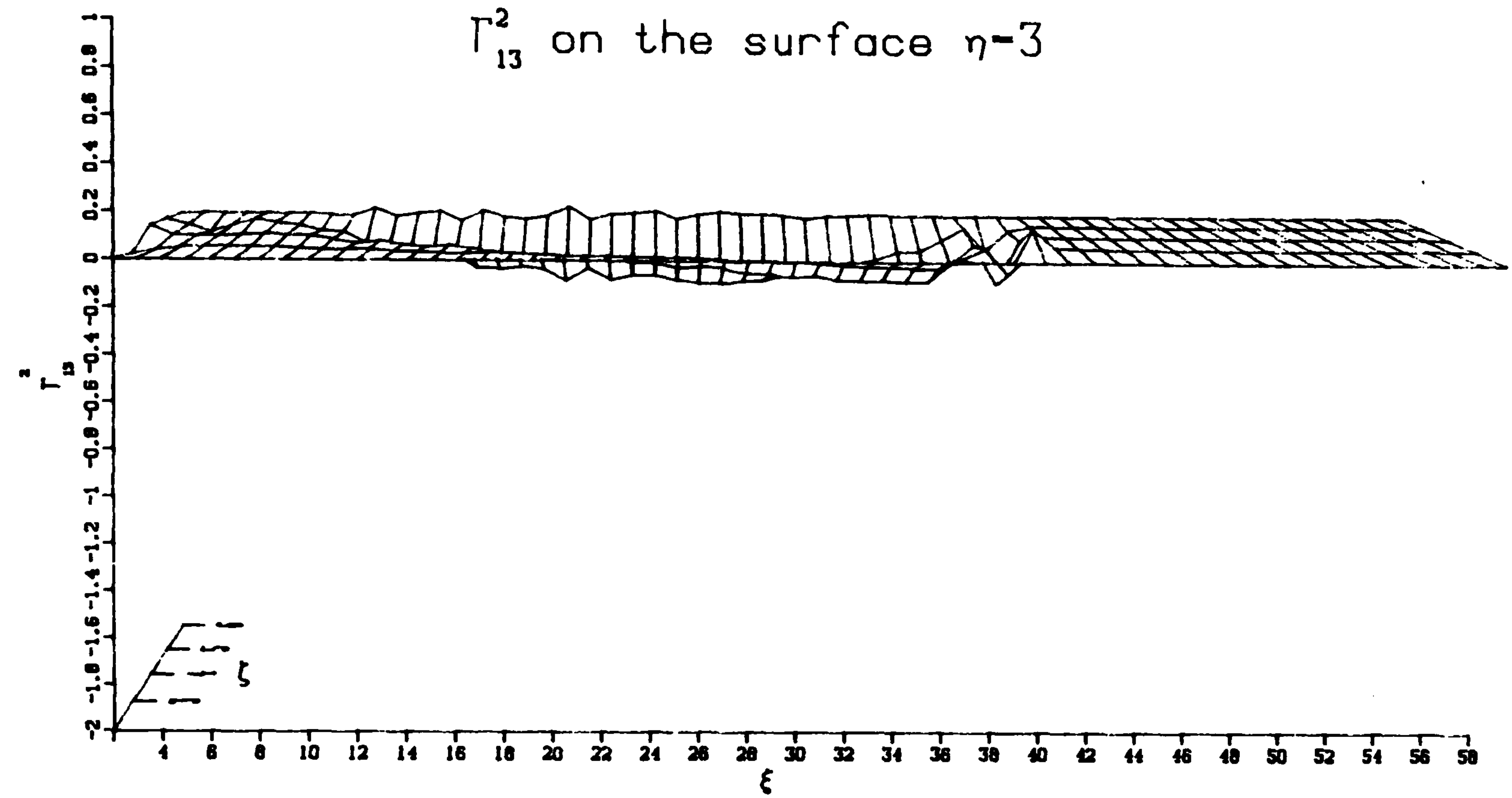

Figure 6. Continued
(b) Γ^2_{13} on the Surface $\eta = 3$.

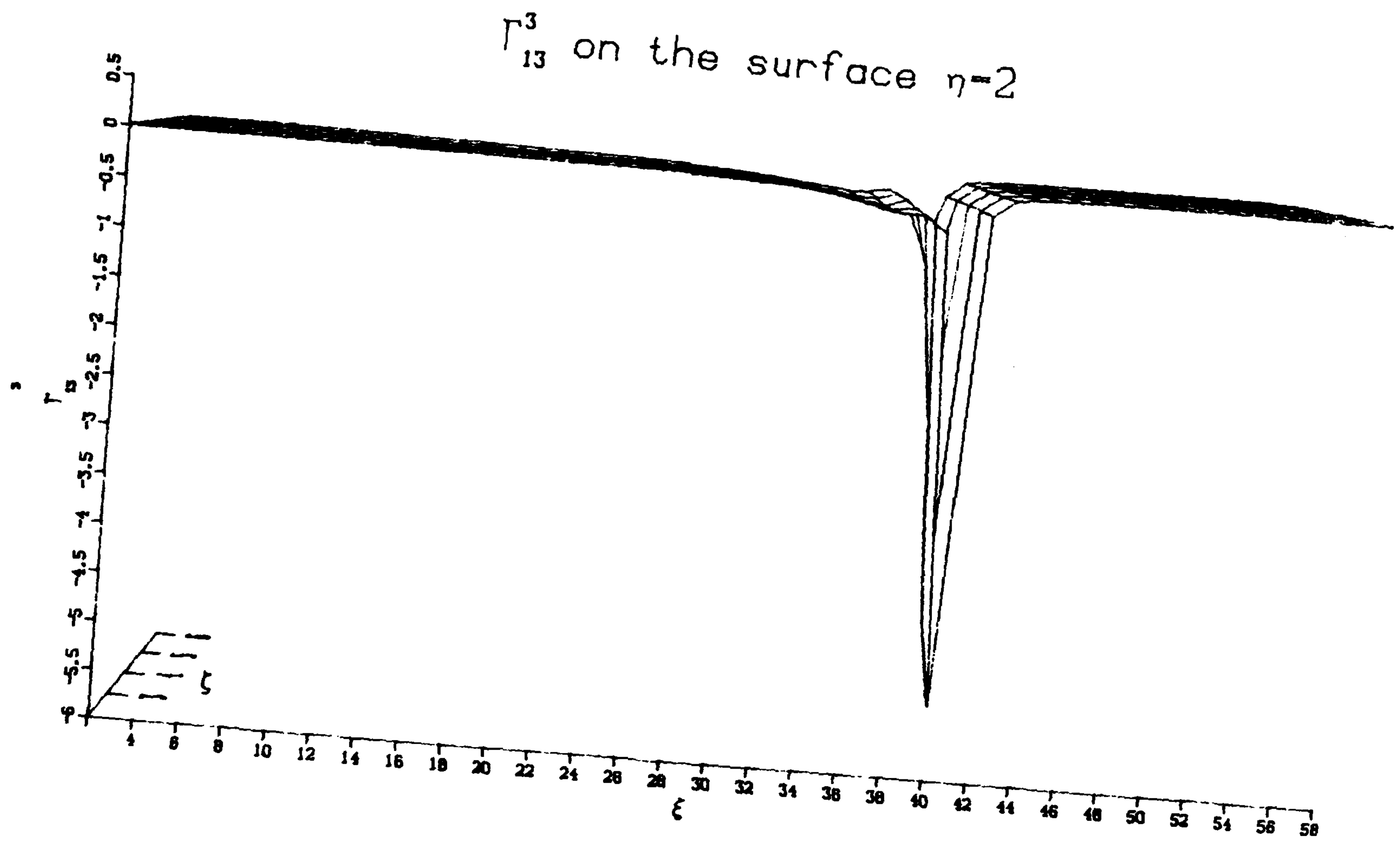

Figure 6. Continued
(c) Γ^3_{13} on the Surface η = 2.

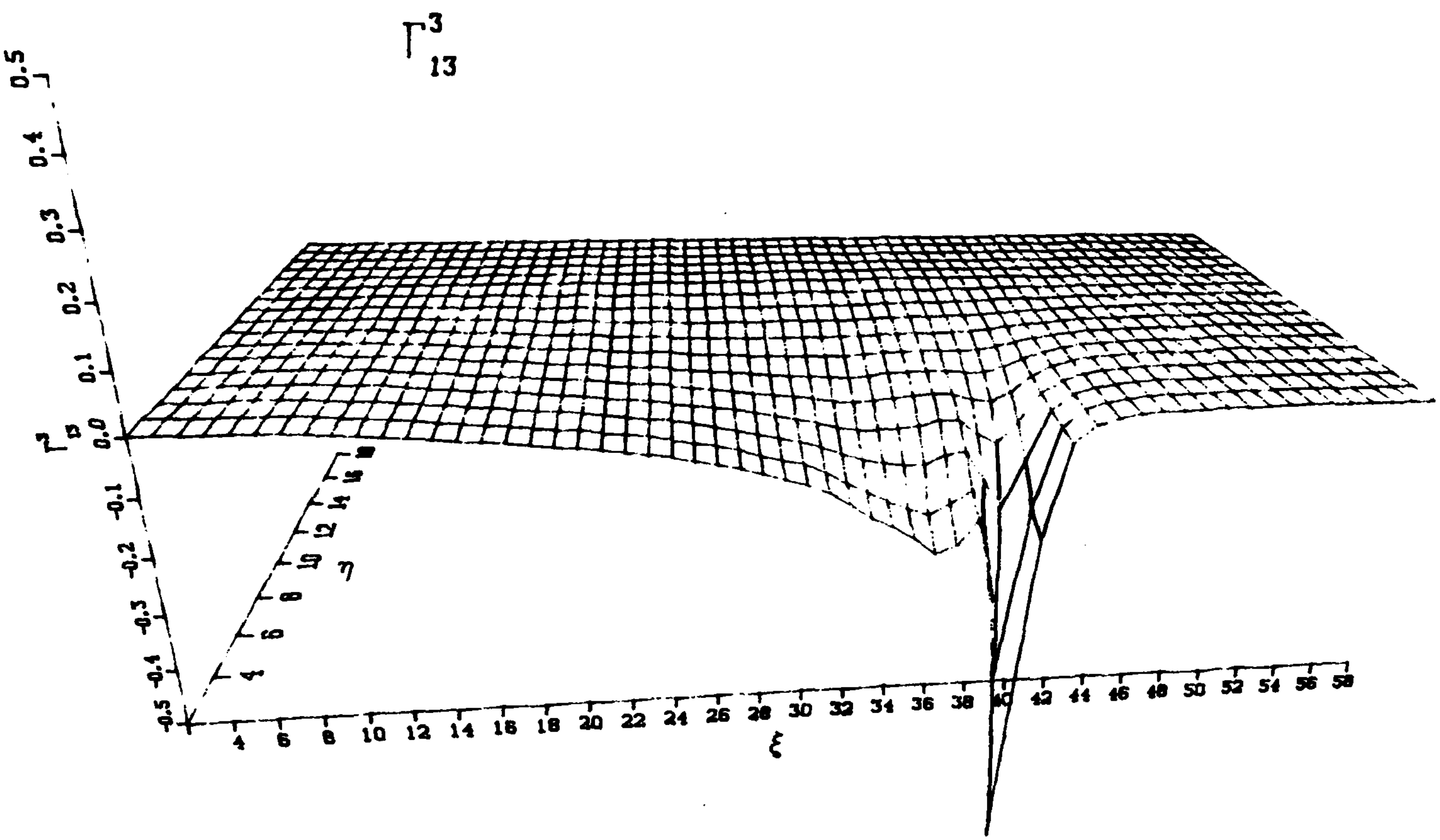

Figure 6. Continued

(d) Γ^{3}_{13} in the Plane $\theta = 90^{\circ}$.

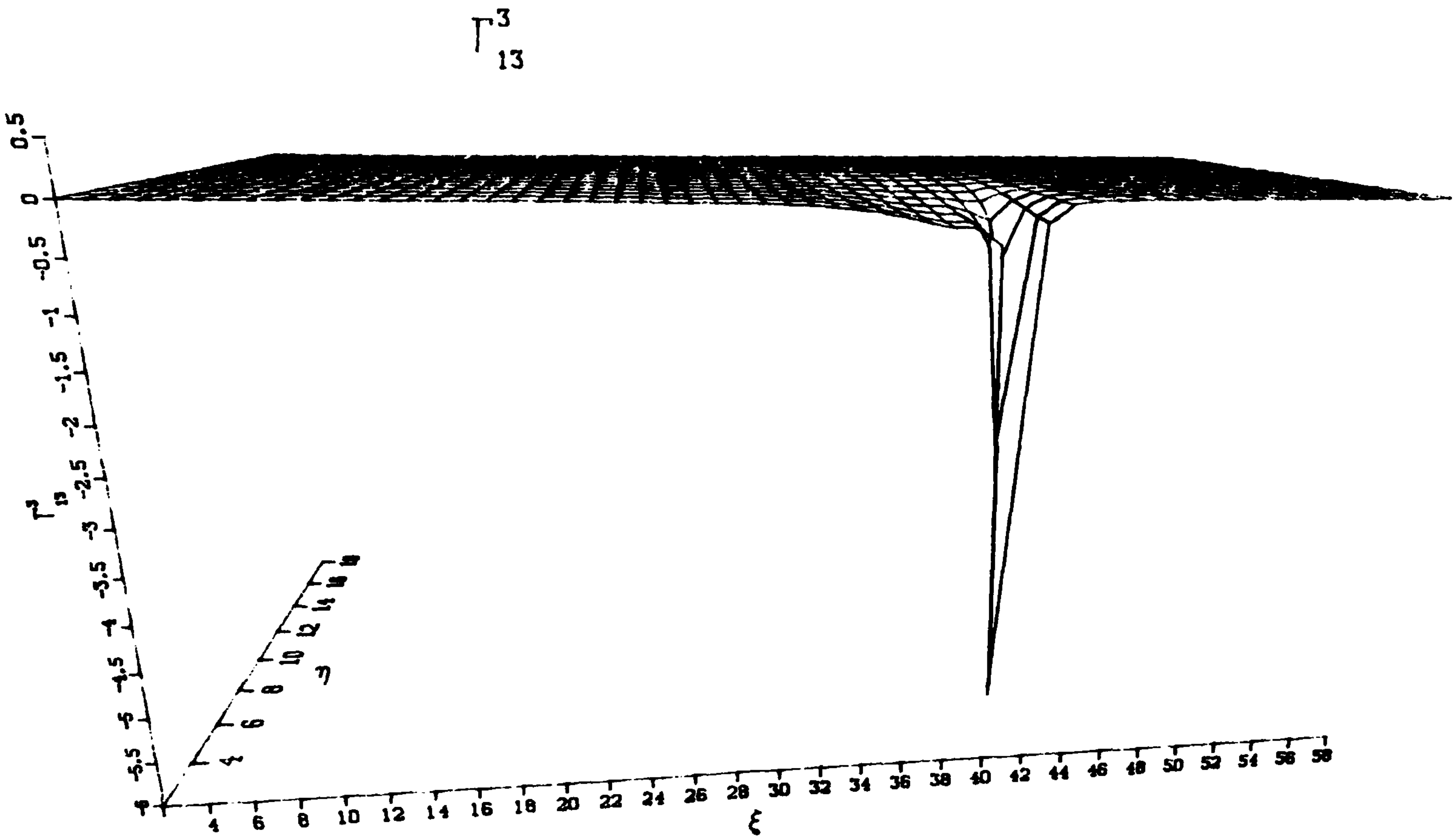

Figure 6. Continued
(e) Γ^3_{13} in the Plane $\theta = 0^o$.

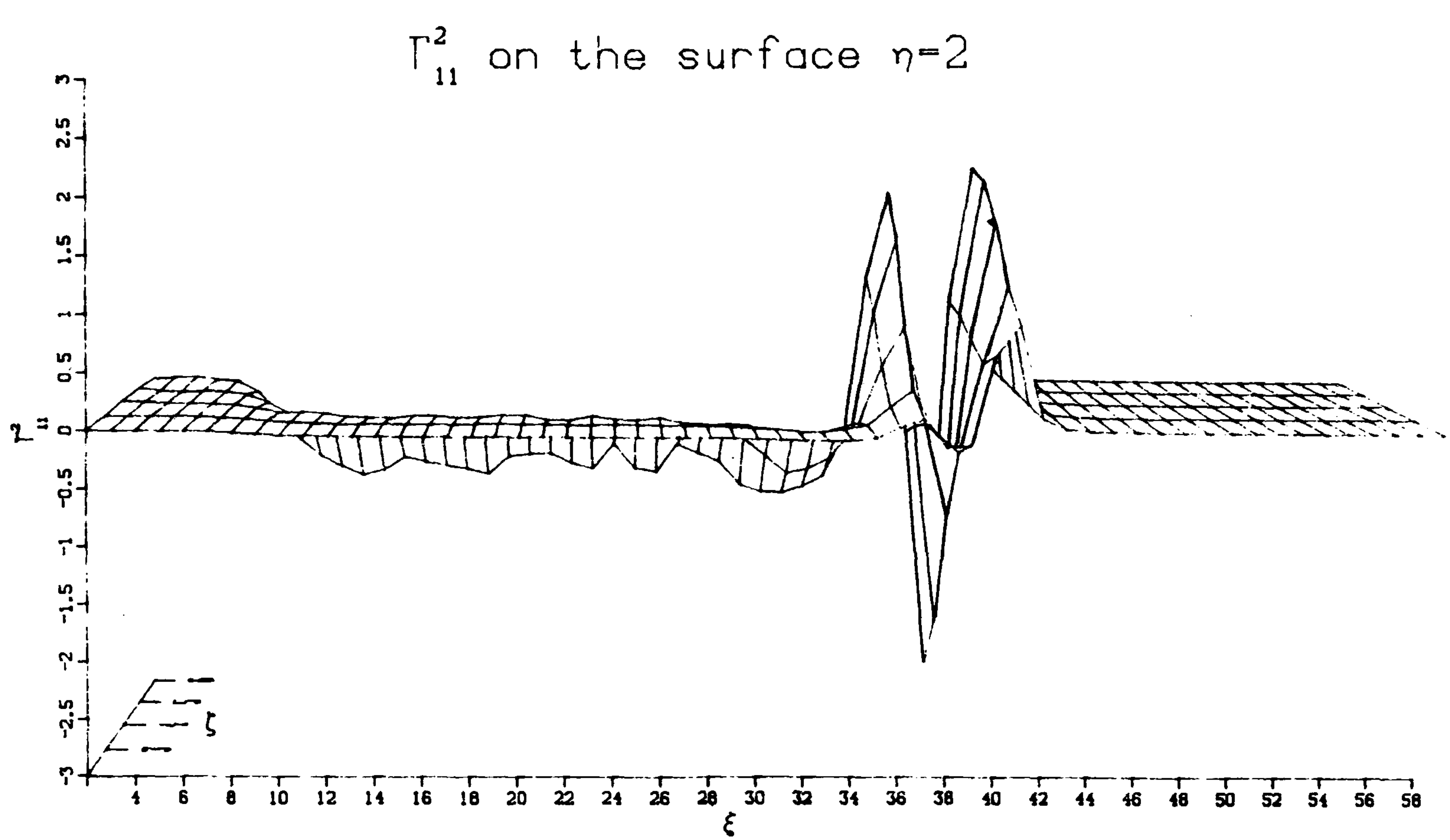

Figure 6. Continued

(f) Γ^2_{11} on the Surface η = 2.

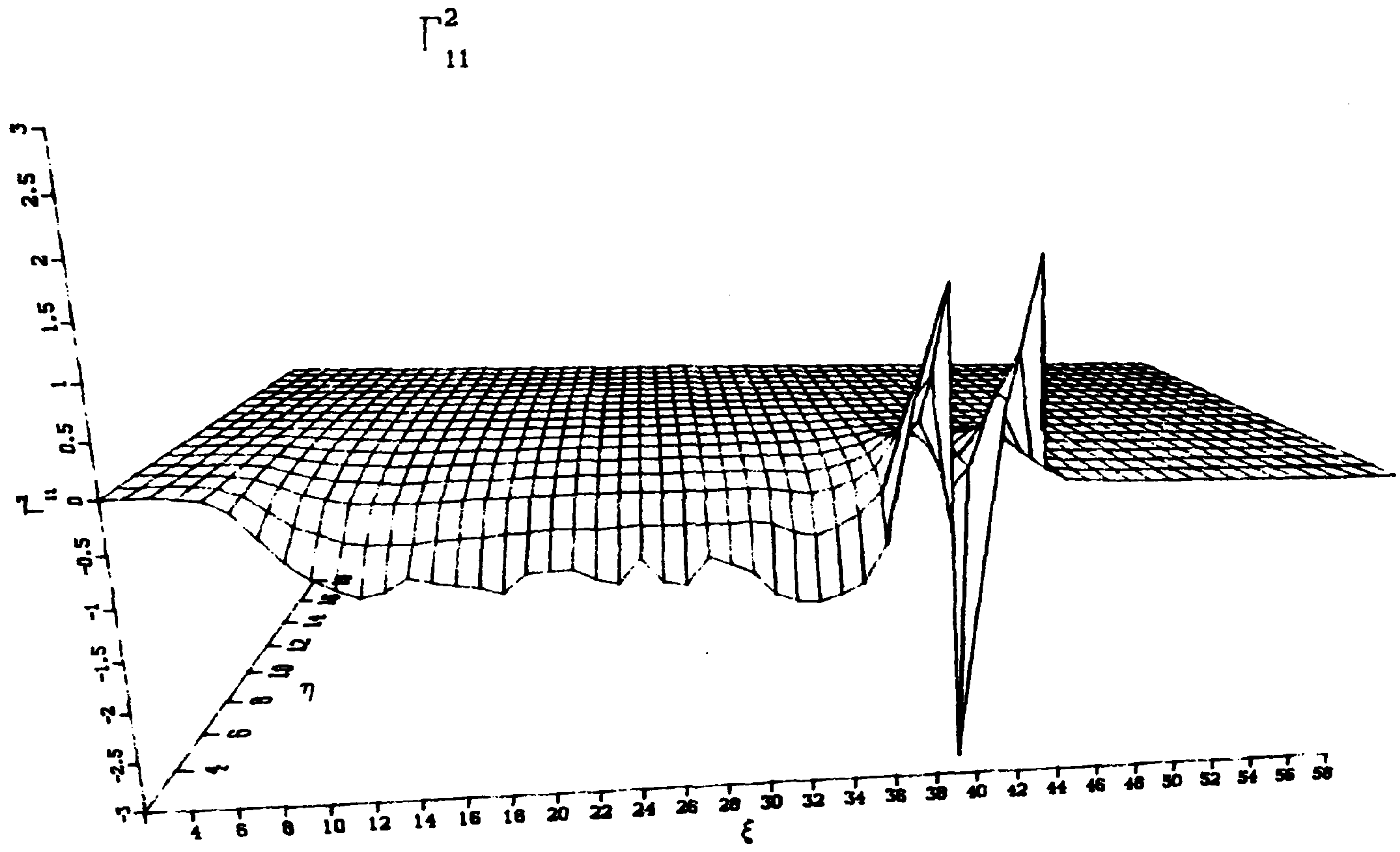

Figure 6. Continued

(g) Γ^2_{11} in the Plane $\theta = 90^\circ$.

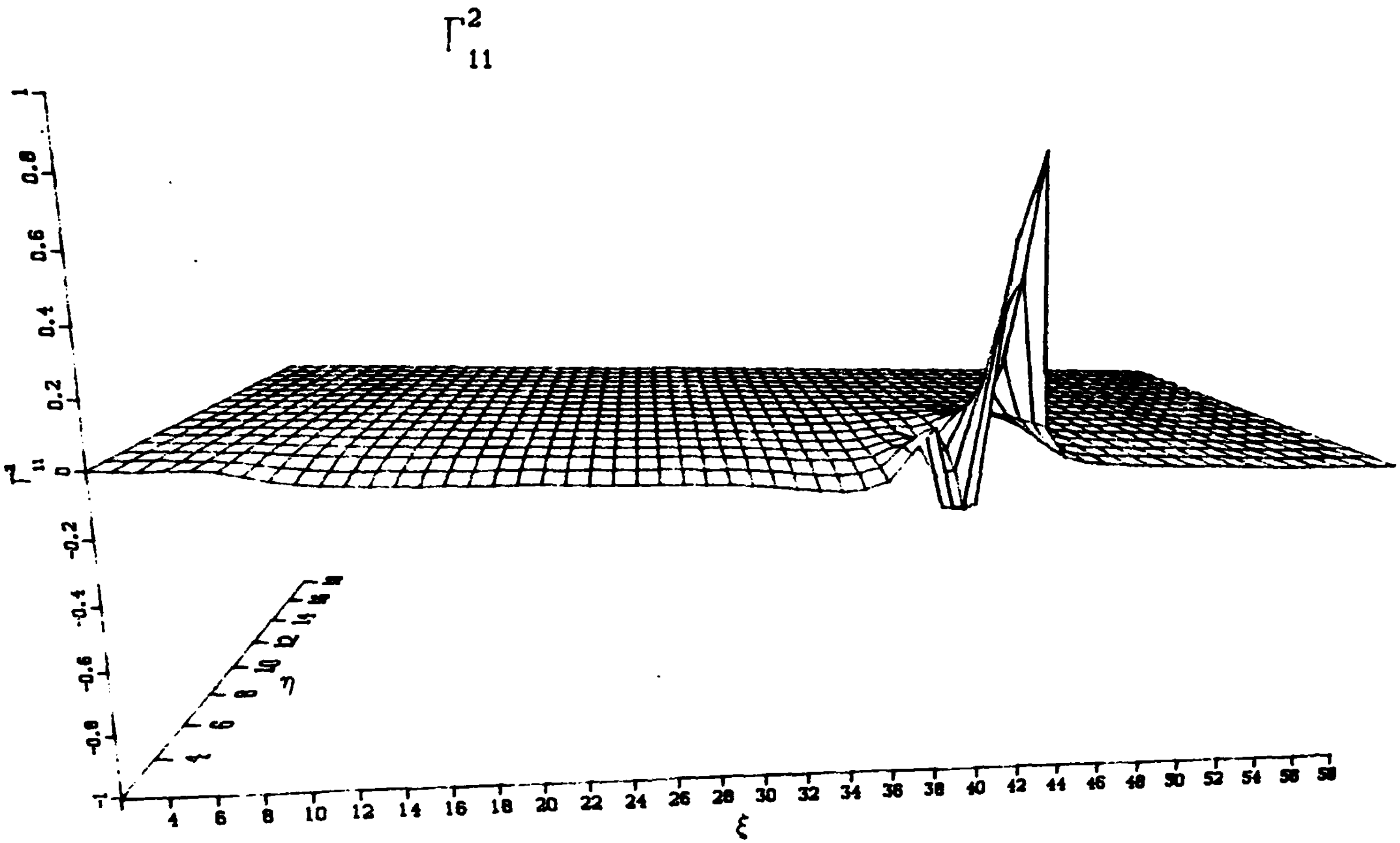

Figure 6. Continued

(h) Γ^2_{11} in the Plane $\theta = 0^o$.

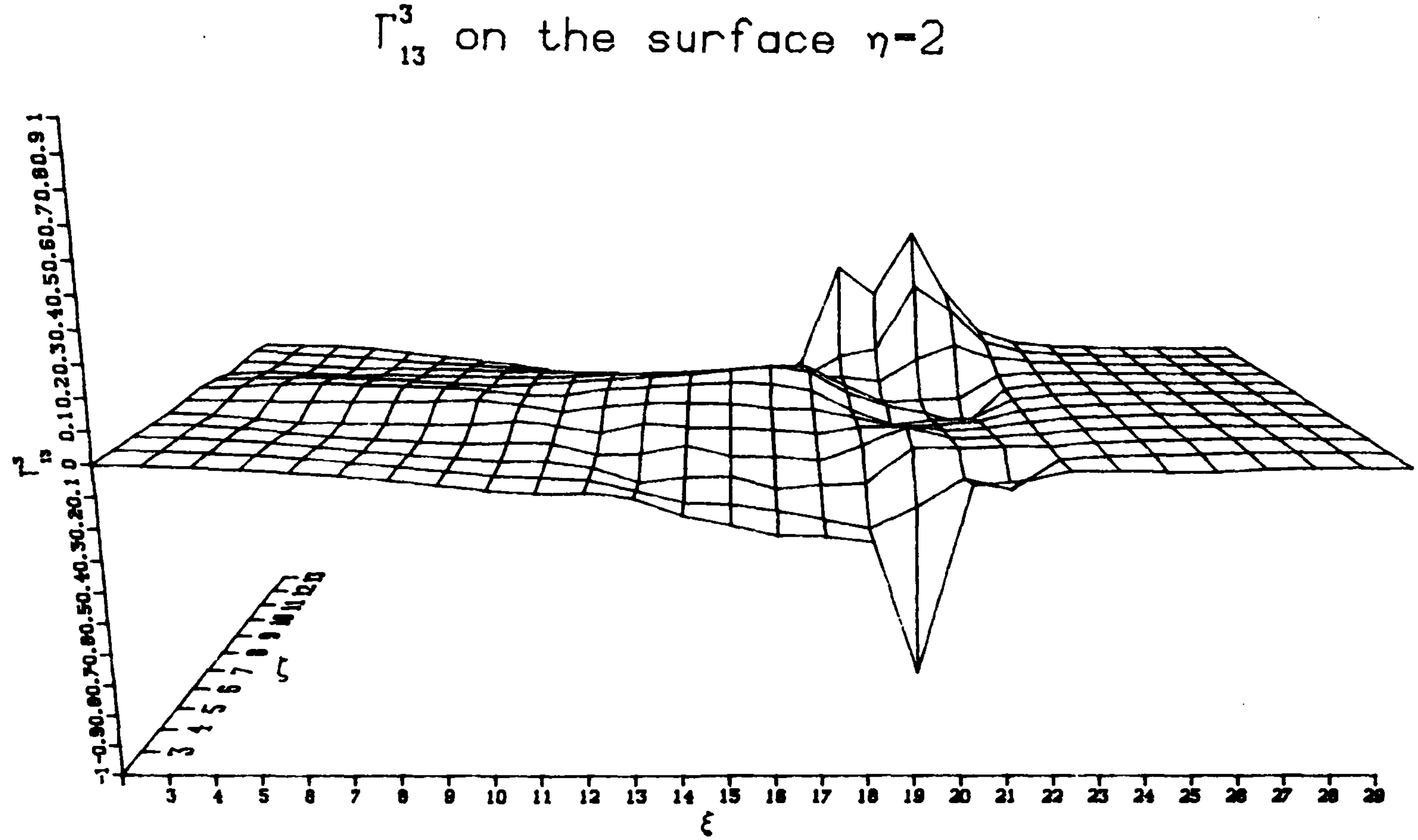

Figure 7. Calculated Christoffel Symbols for the SSPA 720 Liner

(a) Γ^3_{13} on the Surface $\eta = 2$.

Γ^3_{13} along the waterline

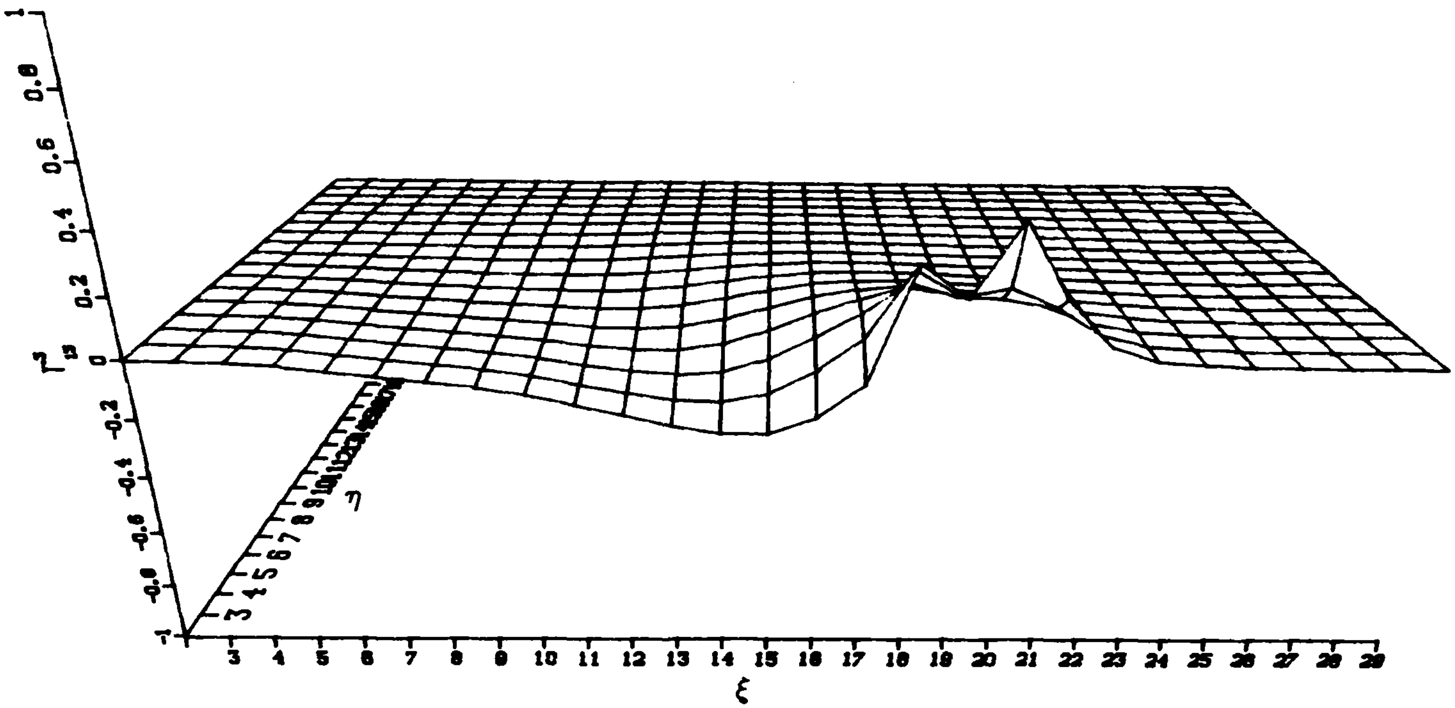

Figure 7. Continued
(b) Γ^3_{13} in the Waterline Plane.

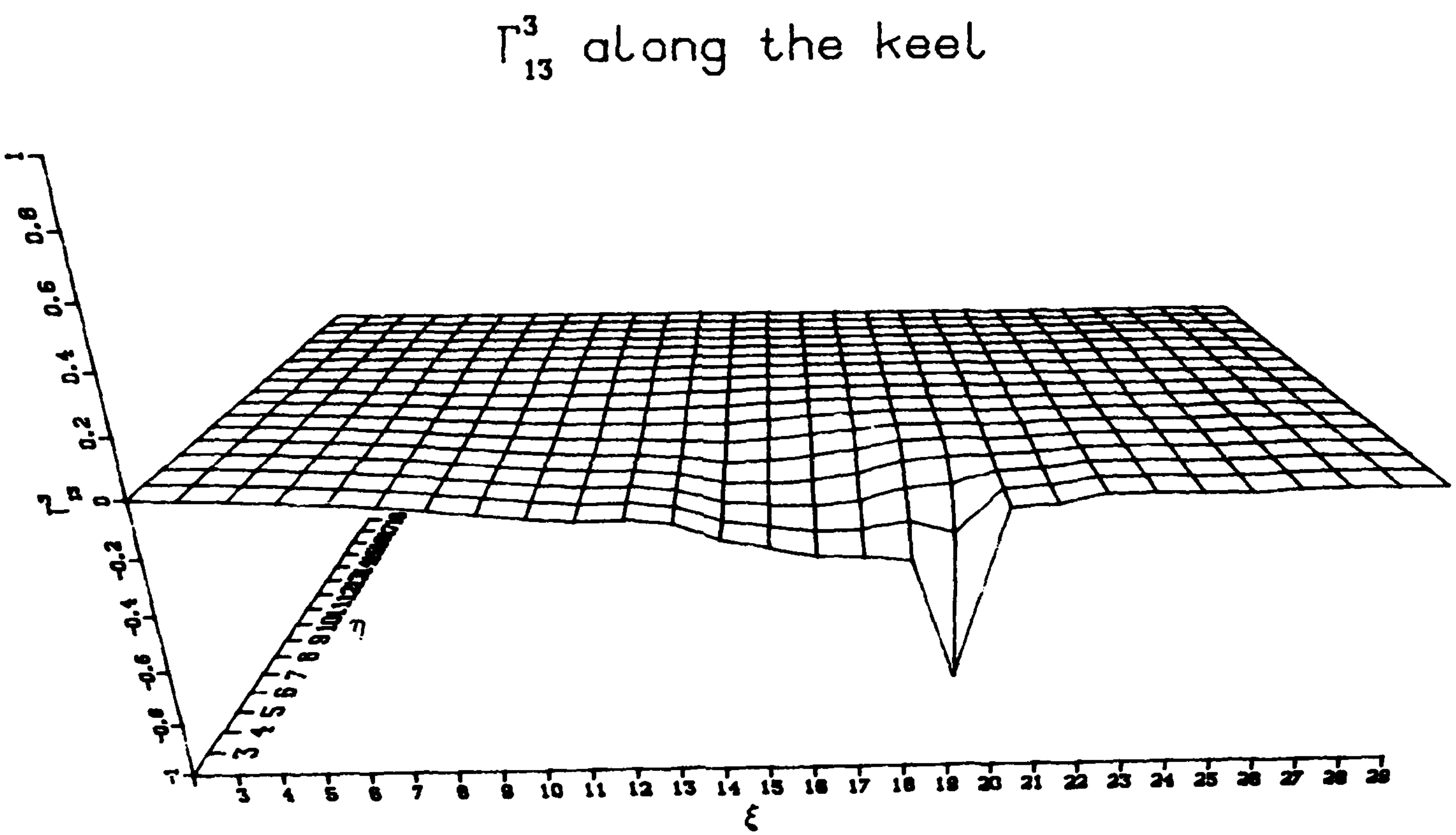

Figure 7. Continued
(c) Γ^3_{13} in the Keel Plane.

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